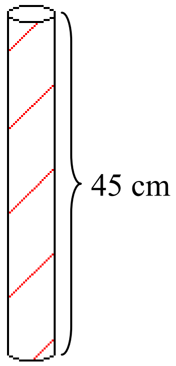
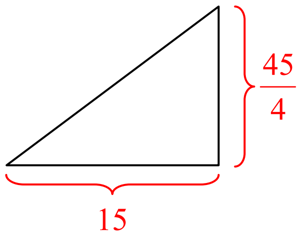
1. A cylinder 45 cm high has a circumference of 15 cm. A string makes exactly 4 complete turns round the cylinder while its two ends touch the cylinder's top and bottom. How long is the string in cm?

Solution

Consider 1 revolution. The height of the new cylinder would be $\frac{1}{4}$ of the height of the original cylinder, i.e. $\frac{45}{4}$ cm. If we cut the cylinder along the line of the string we'd get the following shape:





Let x be the length of the string for 1 revolution. By Pythagoras, $x^2 = 15^2 + \left(\frac{45}{4}\right)^2$. Hence $x = \frac{75}{4}$. For 4 revolutions, the length of the string is $4 \times \frac{75}{4} = 75$ cm.

ANS:75 cm

2. What is the sum of all integers between 500 and 1500 which are divisible neither by 2 nor by 5?

Solution

The sum is

501+503+507+509+511+513+517+519+…+1491+1493+1497+1499

=(501+503+507+509)+(511+513+517+519)+…+(1491+1493+1497+1499)

=2020+2060+2100+…+5980

$= \frac{(2020+5980)\times 100}{2}$

=400000

ANS:400000

3. The symbol $n!$ is used to represent the product $n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$. For example, $4! = 4 \times 3 \times 2 \times 1$. Find n satisfying $n! = 2^{17} \times 3^9 \times 5^4 \times 7^3 \times 11 \times 13 \times 17 \times 19$.

Solution

Since the largest prime factor of $n!$ is 19, n=19, 20, 21 or 22. Since $n!$ has 7^3 as a factor, then n is at least 21. Because 11^2 is not a factor of $n!$ and 11 is a factor of $n!$, n is less than 22. So n=21.

ANS:21

4. In the figure below, the rectangle at the corner measures 3 cm by 6 cm. What is the radius of the circle in cm?

Solution

Using Pythagoras' theorem for the right-angled triangle in the figure alongside,

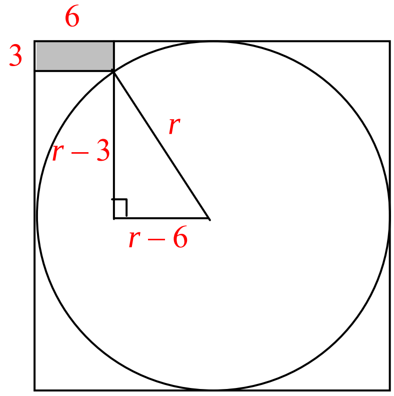
$r^2 = (r-3)^2 + (r-6)^2$

$r^2 - 18r + 45 = 0$

$(r-15)(r-3) = 0$

$r = 15$ or $r = 3$

Clearly $r \neq 3$, so $r = 15$ cm.



ANS:15 cm

5. Divide 2008 marbles into a number of bags so that I can ask for any number of marbles from 1 to 2008, and you can give me the proper amount by giving me a certain number of these bags without opening them. What is the minimum number of bags you will require?

Solution

We can divide 2008 marbles into 11 bags as follows:

$2^0=1$, $2^1=2$, $2^2=4$, $2^3=8$, $2^4=16$, $2^5=32$, $2^6=64$, $2^7=128$, $2^8=256$, $2^9=512$, $2^{10}=1024$.

Thus when we need *n* marbles, we just need to convert *n* from a base-10 integer numeral to its base-2 (binary) equivalent.

ANS:11

6. Let *a* be a real number such that $3a-\frac{3}{a}+1=0$. What is the value of $a^3-\frac{1}{a^3}+3$?

Solution

Since $3a-\frac{3}{a}+1=0$, $3a-\frac{3}{a}=-1$, i.e. $a-\frac{1}{a}=-\frac{1}{3}$. Because $\left(a-\frac{1}{a}\right)^3=a^3-3a+\frac{3}{a}-\frac{1}{a^3}$,

$a^3-\frac{1}{a^3}=\left(a-\frac{1}{a}\right)^3+3a-\frac{3}{a}=-\frac{1}{27}-1=-\frac{28}{27}$. Hence $a^3-\frac{1}{a^3}+3=\frac{53}{27}$.

ANS: $\frac{53}{27}$

7. In an office, there are 14 desks of four types: one-drawer, two-drawer, three-drawer and four-drawer respectively. There are 33 drawers altogether in those desks. How many one-drawer desks are there, if it is known that there are as many of them as the two-drawer and three-drawer desks altogether?

Solution

Let *w*, *x*, *y* and *z* be the numbers of one-drawer, two-drawer, three-drawer and four-drawer desks, respectively. Then we have

$$\begin{cases} w+x+y+z=14 & (1) \\ w+2x+3y+4z=33 & (2) \\ w=x+y & (3) \end{cases}$$

From (1) and (3), we obtain $2w+z=14$. Hence $z=14-2w$. Therefore

$$\begin{aligned} w+2x+3y+4z &= w+2(x+y)+y+4z \\ &= w+2w+y+4(14-2w) \\ &= 56+y-5w \end{aligned}$$

Hence $56+y-5w=33$ in view of (2). Hence $y=5w-23$.

Since $y>0$, we have $w\ge 5$. Since $z>0$ and $z=14-2w$, we obtain $w\le 6$. Therefore $w=5$ or $w=6$.

If $w=6$, then $z=14-2w=2$, $y=5w-23=7$ and $x=w-y=-1$ which is impossible.

If $w=5$, then $z=14-2w=4$, $y=5w-23=2$ and $x=w-y=3$. Thus the number of one-drawer desks is 5.

ANS: 5

8. In a circumference a right triangle ΔABC with hypotenuse AB is inscribed. On the longer leg BC is chosen a point D so that $AC = BD$. Find the angle EDC, if E is the midpoint of the arc ACB.

Solution

Since E is the middle of the arc ACB (AB is a diameter of the circle), the triangle ΔAEB is isosceles and right, which means that $AE = BE$ and $\angle EAB = 45°$. $\angle CAE = \angle DBE$, since they are inscribed and responding to one and the same arc. Therefore $\Delta ECA \cong \Delta EDB$, which means that $CE = DE$. Considering the fact that $\angle ECB = \angle EAB = 45^o$ we obtain that the triangle ΔCED is isosceles with two angles of $45°$. Hence the angle at the vertex E is right i.e. $\angle DEC = 90°$.

Since $CE=ED$, we have $\angle EDC = 45°$