1. Show that, for all positive real numbers p, q, r, s,

$$\left(p^2+p+1\right)\left(q^2+q+1\right)\left(r^2+r+1\right)\left(s^2+s+1\right)\geq 81pqrs$$

Solution

Since $(p-1)^2>0$, $p^2-2p+1>0$ **(5 points) .** i.e. $p^2+p+1>3p$. In the similar manner, we also get $q^2+q+1>3q$, $r^2+r+1>3r$ and $s^2+s+1>3s$ **(10 points)**. Hence

$$\left(p^2+p+1\right)\left(q^2+q+1\right)\left(r^2+r+1\right)\left(s^2+s+1\right)\geq 81pqrs \textbf{ (5 points)}$$

2. In a troop of 2008, 12 are on patrol duty every night. Prove that it is impossible to draw up a schedule according to which every 2 are on duty together exactly once.

Solution

Consider a particular trooper and the nights when he is on patrol duty. Then other 2007 serve with him 11 at a time**(10 points)**. Since 2007 is not divisible by 11, the desired schedule is impossible**(10 points)**.

3. Find all integers that satisfy the equation:

$$x^2-2xy+2x-y+1=0.$$

Solution

We transform the given equation the following way:

$x^2-2xy+2x-y+1=0 \Leftrightarrow y(2x+1)=x^2+2x+1 \Leftrightarrow y=\dfrac{x^2}{2x+1}+1$, so that $2x+1\neq 0$ for every integer x.

y is an integer if and only if $\dfrac{x^2}{2x+1}$ is an integer as well**(5 points)**.

Considering the fact that GCD ($2x$+1, x) = GCD (x+1, x) = 1, we get that x^2 and $2x+1$ are relatively prime **(5 points)**. Hence $\dfrac{x^2}{2x+1}$ is an integer as $2x+1=\pm 1$. **(5 points)** This way we get that the solutions of the equation are (x, y) = (0, 1) and (-1, 0) **(5 points)**.

ANS:(0, 1) and (-1, 0)

4. Markers are to be placed in some squares of a 4×4 chessboard.
   (a) Place 7 markers so that if the markers on any two rows and any two columns are removed, at least one marker remains on the board.
   (b) Prove that no matter how 6 markers are placed on the board, then it is always possible to choose two rows and two columns so that no markers remain on the board when all markers in these rows and columns are removed.

Solution

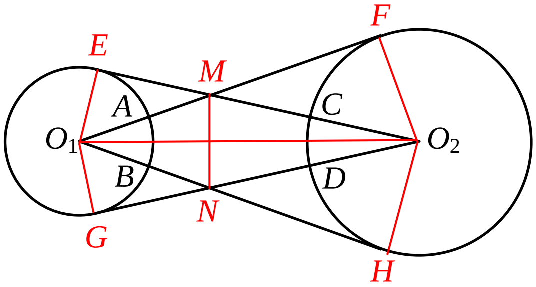
(a)

○	○		
○		○	
	○	○	
			○

(10 points)

(b) If there are at most six markers, consider the two columns which contain the fewest markers between them. If the total is at least three, then one of these two columns must contain at least two markers, whereas one the other two columns must contain at most one, contradicting the minimally assumption of our pair. Hence the total is at most two. **(5 points)** Cross out the other two columns, and we have at most two markers left. We can now cross out at most two rows in order to remove any remaining markers. **(5 points)**

5. From the centers of two “exterior” circles draw the tangents to the other circle. Prove that *AB*=*CD*.



Solution

Since $\angle EMO_1 = \angle FMO_2$ and $\angle MEO_1 = \angle MFO_2 = 90°$, $\Delta MEO_1 \sim \Delta MFO_2$ and hence $\frac{EO_1}{MO_1} = \frac{FO_2}{MO_2}$. Since $EO_1 = AO_1$ and $FO_2 = CO_2$, we have $\frac{AO_1}{MO_1} = \frac{CO_2}{MO_2}$ **(5 points)**.

Since $\angle EO_1M = \angle EO_1O_2 - \angle MO_1O_2 = \angle GO_1O_2 - \angle NO_1O_2 = \angle GO_1N$, $EO_1 = GO_1$ and $\angle MEO_1 = \angle MFO_2 = 90°$, we get $\Delta MEO_1 \cong \Delta NGO_1$ and hence $MO_1 = NO_1$.

So $\Delta MNO_1 \sim \Delta ABO_1$ and hence $\frac{AO_1}{MO_1} = \frac{AB}{MN}$ **(10 points)**.

In the similar manner, we also get $\Delta MNO_2 \sim \Delta CDO_2$ and hence $\frac{CO_2}{MO_2} = \frac{CD}{MN}$ **(5 points)**. So we have $\frac{AB}{MN} = \frac{AO_1}{MO_1} = \frac{CO_2}{MO_2} = \frac{CD}{MN}$, i.e. *AB*=*CD*.

6. Find all possible integers *N* satisfying the following properties:
   (i) *N* has at least two prime divisors, and
   (ii) $N = d_1^2 + d_2^2 + d_3^2 + d_4^2$, where d_1, d_2, d_3 and d_4 are the first four positive divisors of *N*.

Solution

Clearly $d_1 = 1$ **(5 points)**. If *N* is an odd number, then d_2, d_3 and d_4 are odd numbers. This result implies that $N = d_1^2 + d_2^2 + d_3^2 + d_4^2$ is even which contradict to our assumption. So *N* is even and hence $d_2 = 2$ **(5 points)**. Observe that one of d_3 and d_4 is an odd number. So there are two possibilities: $d_3 = 4$ or d_3 is an odd prime. If $d_3 = 4$, then $4 | N$ and d_4 is odd. Thus $N = d_1^2 + d_2^2 + d_3^2 + d_4^2 \equiv 2 (\text{mod } 4)$ which contradict to $4 | N$. Hence d_3 is an odd prime. So $d_4 = 2d_3$ **(5 points)**. Then $N = d_1^2 + d_2^2 + d_3^2 + d_4^2 = 1 + 4 + 5d_3^2 = 5 + 5d_3^2 = 5(1 + d_3^2)$. Since d_3 and $1 + d_3^2$ are relatively prime, $d_3 = 5$. So $N = 5(1 + 5^2) = 130$ **(5 points)**.

ANS: 130