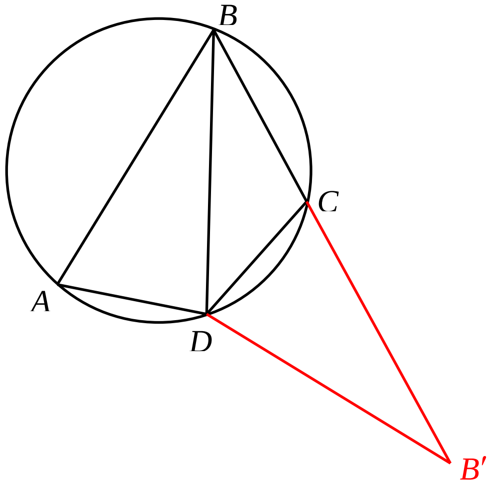
1. The diagonal BD of the inscribed quadrilateral $ABCD$ is the bisector of $\angle ABC$. Find the area of the quadrilateral if $BD = 10$ cm and $\angle ABC = 60^\circ$.

Solution



Since on the equal arcs of a circumference correspond equal chords then in the given quadrilateral $ABCD$, we know $AD = CD$. We consider rotation with D as a centre and the angle being $\angle ADC$ clockwise. In this rotation point A and B transform into C and B' respectively, so ΔABD transforms into its identical triangle $\Delta CB'D$. Since $ABCD$ is inscribed, so $\angle BCD + \angle BAD = 180^\circ$, i.e. $\angle BCD + \angle DCB' = 180^\circ$, which means points C, B and B' lie on the same line. This means that $ABCD$ and $\Delta BDB'$ have equal areas. Since $BD = 10$ cm and $\angle DBC = 30^\circ$, the altitude on BB' is 5 cm and $BB' = 5\sqrt{3}\times 2 = 10\sqrt{3}$ cm. Then the area is $\frac{1}{2}\times 5\times 10\sqrt{3} = 25\sqrt{3}$ cm^2.

ANS: $25\sqrt{3}$ cm^2

2. The function $f(x)$ satisfies the equation

$$f(2008^x) + xf(2008^{-x}) = 2008$$

for all values of x. What is the value of $f(2008)$?

Solution

For x=1, we have $f(2008) + f(2008^{-1}) = 2008$.

For x=-1, we have $f(2008^{-1}) - f(2008) = 2008$.

Hence $f(2008) = 0$.

ANS:0

3. Let a be the sum of the digits of an arbitrary 2008-digit multiple of 9. Let b be the sum of the digits of a, and c be the sum of the digits of b. Determine c.

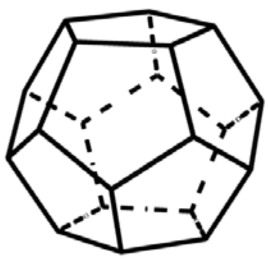
Solution

Since a is the sum of the digits of an arbitrary 2008-digit multiple of 9, a is a multiple of 9 and hence b and c are multiples of 9. Because $a \le 2008\times 9 = 18072$, $b \le 1+7+9+9+9 = 35$. So b=9, 18 or 27 and hence c=9.

ANS:9

4. A desk calendar consists of a regular dodecahedron with a different month on each of its twelve pentagonal faces. How many essentially different ways are there of arranging the months on the faces?

Solution



Pick any face for January. There are C_5^{11} ways of choosing the months to go into the ring of five faces adjacent to January, and 4! essentially different ways of arranging them. There is a second ring of five faces, each adjacent to two of January's neighbors; the months for these can be chosen in C_5^6 ways, and there are 5! essentially different ways of arranging them relative to the first ring. Finally, the month for the face antipodal to January's face is now determined. Hence the number of essentially different ways of making the calendar is $C_5^{11}\times 4!\times C_5^6\times 5! = \frac{11!}{5}$ (= 7983360).

ANS: $\frac{11!}{5}$ or 7983360

5. Find an integer x satisfying the following equation.

$$\left[\frac{x}{1!}\right]+\left[\frac{x}{2!}\right]+\left[\frac{x}{3!}\right]+\cdots+\left[\frac{x}{10!}\right]=2008,$$

where $[x]$ denotes the greatest integer which is less than or equal to x.

Solution

Since $\left[\frac{x}{1!}\right]\le 2008<5040=7!$, we see $x<7!$. Let $x=a\times 6!+b\times 5!+c\times 4!+d\times 3!+e\times 2!+f$,

($a\le 6, b\le 5, c\le 4, d\le 3, e\le 2,$ and $f\le 1$). Then, $2008=1237a+206b+41c+10d+3e+f$.

Since $206b+41c+10d+3e+f\le 206\times 5+41\times 4+10\times 3+3\times 2+1=1231$, $777\le 1237a\le 2008$.

So $a=1$ and hence $771=206b+41c+10d+3e+f$.

Since $41c+10d+3e+f\le 41\times 4+10\times 3+3\times 2+1=201$, $570\le 206a\le 771$. So $b=3$ and hence $153=41c+10d+3e+f$.

Since $10d+3e+f\le 10\times 3+3\times 2+1=37$, $116\le 41c\le 153$. So $c=3$ and hence $30=10d+3e+f$.

Since $3e+f\le 3\times 2+1=7$, $23\le 10d\le 30$. So $d=3$ and hence $e=f=0$.

So $x=1\times 6!+3\times 5!+3\times 4!+3\times 3!=1170$

ANS: 1170

6. It's given that $4x^2+25y^2+196z^2=144$. Find the maximal possible value of $4x+5y-28z$.

Solution

Let's have a look at the vectors $\vec{m}=(2x,5y,14z)$ and $\vec{n}=(2,1,-2)$ so that $\vec{m}\vec{n}=4x+5y-28z$. Then $|\vec{m}|=\sqrt{4x^2+25y^2+196z^2}=12, |\vec{n}|=3$. As $\vec{m}\vec{n}\le|\vec{m}|\times|\vec{n}|$, then $4x+5y-28z\le 36$.

We have equality when and only when $\vec{m}//\vec{n}$, meaning:

$$\frac{2x}{2}=\frac{5y}{1}=\frac{14z}{-2}\Leftrightarrow\begin{cases}x=-7z\\ y=-\frac{14}{5}z\end{cases}$$

As substituting in the given equation we obtain that the value is reached when $(x,y,z)=\left(4,\frac{4}{5},-\frac{4}{7}\right)$.

ANS: 36

7. Determine a constant k such that the polynomial

$$P(x,y,z)=x^5+y^5+z^5+k(x^3+y^3+z^3)(x^2+y^2+z^2)$$

has the factor $x+y+z$.

Solution

If we think of y and z as fixed and x as the variable, so that $P(x,y,z)$ is a polynomial in x, we see that : $x+y+z=x-(-y-z)$ is a factor of $P(x,y,z)$ if and only if $P(-y-z,y,z)=0$. Thus, we seek values of k for which $P(-y-z,y,z)=0$. This identity is

$$(-y-z)^5+y^5+z^5+k((-y-z)^3+y^3+z^3)((-y-z)^2+y^2+z^2)=0$$

Simplifying, we obtained

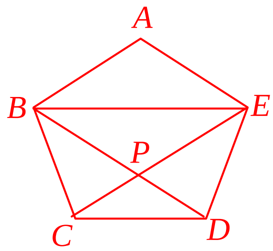
$$-(5+6k)yz(y+z)(y^2+yz+z^2)=0,$$

So that $k=-\frac{5}{6}$.

ANS: $-\frac{5}{6}$

8. A given convex pentagon *ABCDE* has the property that the area of each of the five triangles *ABC*, *BCD*, *CDE*, *DEA* and *EAB* is 1. Find the area of the pentagon.

Solution



Let S_{ABC} denote the area of $\triangle ABC$ etc. Since $S_{EDC} = S_{BDC} = 1$, both of $\triangle BDC$ and $\triangle EDC$ have equal altitudes on side *CD*. Hence *CD*//*BE*. Similarly, the other diagonals are parallel to their opposite side. Thus *ABPE* is a parallelogram and hence $S_{PBE} = 1$.

Let $S_{PCD} = x$ and hence $S_{BPC} = S_{EPD} = 1 - x$. Since $\frac{S_{PBE}}{S_{BPC}} = \frac{PE}{PC} = \frac{S_{EPD}}{S_{PCD}}$,

$\frac{1}{1-x} = \frac{1-x}{x}$, i.e. $x^2 - 3x + 1 = 0$. So $x = \frac{3-\sqrt{5}}{2}$ and the area of the pentagon is

$1 + 1 + 2(1-x) + x = 4 - x = \frac{5+\sqrt{5}}{2}$

ANS: $\frac{5+\sqrt{5}}{2}$