1. Prove that $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$, where n is a positive integer.

Solution

Since $\left(\sqrt{n}+\sqrt{n+1}\right)^2 = n + 2\sqrt{n^2+n} + n + 1 > n + 2\sqrt{n^2} + n + 1 = 4n+1 = \left(\sqrt{4n+1}\right)^2$, we get $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1}$. **(10 points)**

Because $\left(\sqrt{n+1}-\sqrt{n}\right)^2 > 0$, $2n+1 > 2\sqrt{n(n+1)}$. **(5 points)**

Hence $\left(\sqrt{n}+\sqrt{n+1}\right)^2 = n + 2\sqrt{n(n+1)} + n + 1 < n + (2n+1) + n + 1 = 4n+2 = \left(\sqrt{4n+2}\right)^2$. So we get $\sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$. **(5 points)**

Thus $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$.

2. Two players take turns choosing one number at a time (without replacement) from the set $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. The first player to obtain three numbers (out of three, four, or five) which sum to 0 wins. Does either player have a forced win?

Solution

Consider a 3×3 magic square, wherein all of the rows, columns, and diagonals sum to 0 **(5 points)**; example below. It's not difficult to see that the aim of the game, as stated, can be satisfied if and only if, the three integers fall in the same row, column, or diagonal.

1	2	-3
-4	0	4
3	-2	-1

Hence the game is equivalent to tic-tac-toe, **(10 points for giving an example)** a game which, with best play, is well known to be a draw. Therefore neither player has a forced win. **(5 points)**

3. Let n be a positive integer and x_1, x_2, …, x_{n-1} and x_n be integers such that

$$x_1 + x_2 + \cdots + x_n = 0 \quad \text{and} \quad x_1 x_2 \cdots x_n = n.$$

Prove that n is a multiple of 4.

Solution

Assume that n is an odd integer.

Since $x_1 x_2 x_3 \cdots x_n = n$ is odd, $x_1, x_2, \cdots,$ and x_n are odd integers. However, $x_1 + x_2 + \cdots + x_n$ is odd, so $x_1 + x_2 + \cdots + x_n \neq 0$ which contradicts with $x_1 + x_2 + \cdots + x_n = 0$.

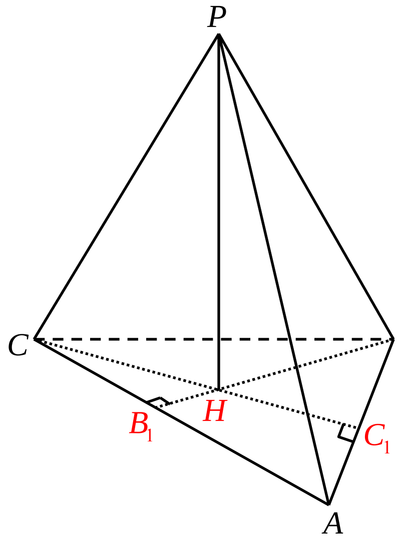
Hence n is an even integer. **(10 points)**

Since n is even, $\exists i \in [1, n]$ such that x_i is even. **(5 points)**

If $x_1, x_2, \cdots, x_{i-1}, x_{i+1} \cdots,$ and x_n are odd, then $x_1 + x_2 + \cdots + x_n$ is odd, which contradicts to the hypothesis $x_1 + x_2 + \cdots x_n = 0$.

Hence $\exists x_j \,(j \in [1, n], j \neq i)$ such that x_j is even. Therefore $n = x_1 x_2 x_3 \cdots x_n$ is divisible by 4. **(5 points)**

4. In the tetrahedron $PABC$ the altitude from P passes through the orthocenter of triangle ΔABC. Find the ratio of the areas of triangles ΔPAB and ΔPAC if $PC=6-\sqrt{2}$, $PB=6+\sqrt{2}$, $BC=2\sqrt{19}$.



Solution

Let $PABC$ be the given tetrahedron, BB_1 and CC_1 be the altitudes of the triangle ΔABC and H is its orthocenter.
Since $PB^2+PC^2=(6+\sqrt{2})^2+(6-\sqrt{2})^2=76=4\times 19=BC^2$, which means triangle ΔPBC is a right triangle (by the Pythagorean Theorem). **(5 points)**
Since the line CC_1 is an orthogonal projection of the line PC on the plane ABC and $CC_1\perp AB$, then $PC\perp AB$ (by the three perpendiculars theorem). **(5 points)** Since $PC\perp PB$, $PC\perp APB$ (due to the sign of perpendicularity of a line and a plane). So $PC\perp PA$. In a similar manner we prove that $PA\perp PB$. **(5 points)**

Hence $\frac{S_{\Delta PAB}}{S_{\Delta PAC}}=\frac{\frac{1}{2}PB\times PA}{\frac{1}{2}PC\times PA}=\frac{PB}{PC}=\frac{6+\sqrt{2}}{6-\sqrt{2}}=\frac{19+6\sqrt{2}}{17}$ **(5 points)**

5. Your calculator is not working properly—it cannot perform multiplications. But it can add (and subtract) and it can compute the reciprocal $\frac{1}{x}$ of any number x. Can you nevertheless use this defective calculator to multiply numbers?

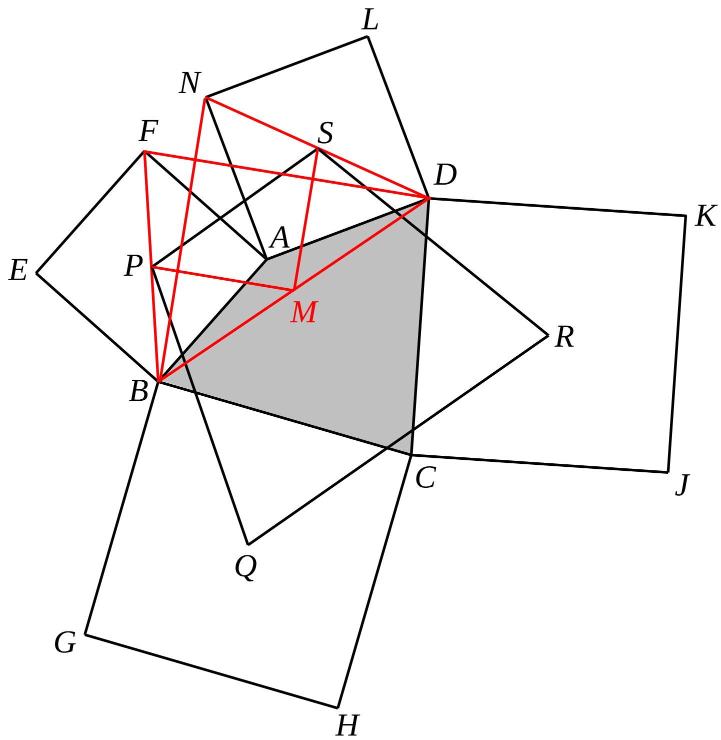
Solution

If $u\neq 1$, we can find u^2 from $u^2=u-\left(\frac{1}{u}+\frac{1}{1-u}\right)^{-1}$ **(10 points)**

and $\frac{1}{2}u$ from $\left(\frac{1}{u}+\frac{1}{u}\right)^{-1}$ **(5 points)**.

Then we can find any product vw from $yw=\left(\frac{v+w}{2}\right)^2-\left(\frac{v-w}{2}\right)^2$ **(5 points)**.

6. Given an arbitrary convex quadrilateral $ABCD$ and the centers P, Q, R, S of the external squares on the sides AB, BC, CD, DA, respectively. Show that the area of quadrilateral $PQRS$ is $\frac{1}{2}\times QS\times QS$



Solution

Let M be the middle point of BD. Connect PM, SM, FD, FB, NB and ND. Since P and S are the centers of square $ABEF$ and $DANL$, P and S are the middle points of FB and ND, respectively. Hence we get:

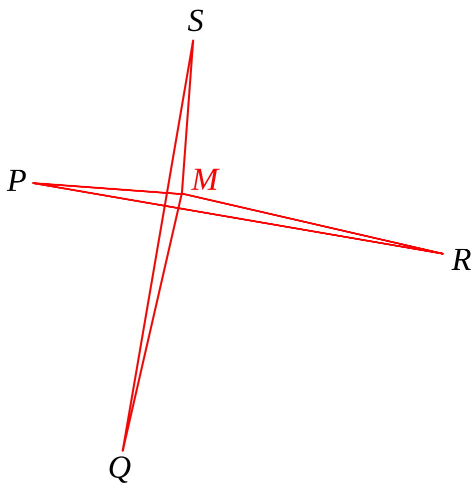
(i) $PM//FD$ and $PM=\frac{1}{2}FD$

(ii) $SM//NB$ and $SM=\frac{1}{2}NB$

(4 points)

Since $AM=AD$, $AB=AF$ and $\angle BAN=\angle FAD$, we also know $\Delta BAN \cong \Delta FAD$ and hence $FD=NB$, i.e. $PM=SM$. **(4 points)**

We consider rotation with A as a centre and the angle being 90° clockwise. In this rotation point B and N transform into F and D, respectively, so $PM \perp SM$. In a similar manner we also get $QM=RM$ and $QM \perp RM$.**(4 points)**



Since $PM=SM$, $RM=QM$ and $\angle SMQ=\angle PMR$, $\Delta SMQ \cong \Delta PMR$ and hence $QS=PR$. **(4 points)**

We consider rotation with M as a centre and the angle being 90° anticlockwise. In this rotation point S and Q transform into P and R, respectively, so $SQ \perp PR$. Hence $PQRS$ is a deltoid and its area is

$\frac{1}{2}\times QS\times PR=\frac{1}{2}\times QS\times QS$ **(4 points)**