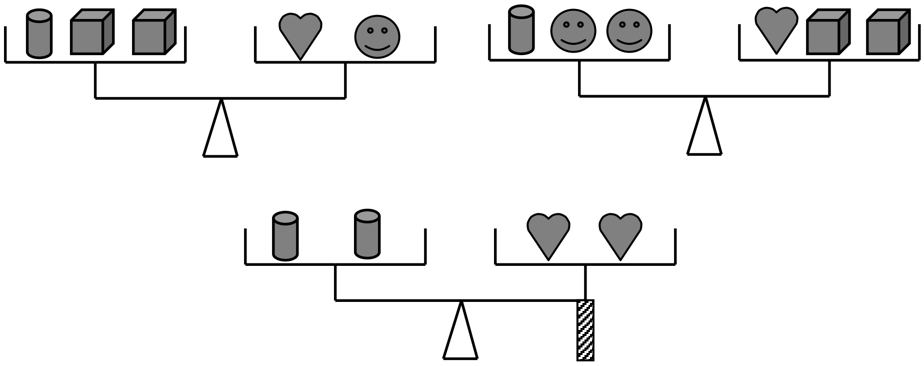
INTERNATIONAL MATHEMATICS AND SCIENCE OLYMPIAD FOR PRIMARY SCHOOLS (IMSO) 2008

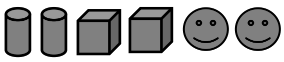
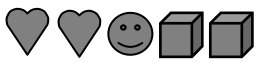
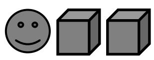
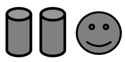
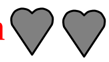
Mathematics Contest in Taiwan

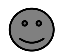
Name:________ School:____________ Grade:_____ number:______

Short Answer: there are 12 questions, fill in the correct answers in the answer sheet. Each correct answer is worth 10 points. Time limit: 90 minutes.

1. In the diagram the top two sets of scales are in perfect balance. For the third set, the right hand side is heavier than the left hand side, and has to be supported as shown. What can be added to the left hand side to achieve a perfect balance in this case as well?



Form the top two sets of scales,  will be in perfect balance with  . Removing  from both side of these collections, we see that  would be in perfect balance with  . It follows that if we add  to the left hand side of the third sets of scales, we achieve a perfect balance.

ANS: 

2. A rectangle is divided into 9 small rectangles, as shown in the diagram, which is not drawn to scale. The areas of 5 of the small rectangles (in suitable units) are given. What is X?

6	9	
	15	18
X		27

If we call the unknown area *A*, *B* and *C* as shown:

6	9	*A*
B	15	18
X	*C*	27

We can use the rule that says that the ratio of the areas of two rectangles having a side in common equals the ratio of their respective side perpendicular to the common side. Thus

$$\frac{B}{15}=\frac{6}{9}, \text{ so } B=10$$

$$\text{and } \frac{A}{9}=\frac{18}{15}, \text{ so } A=\frac{54}{5}$$

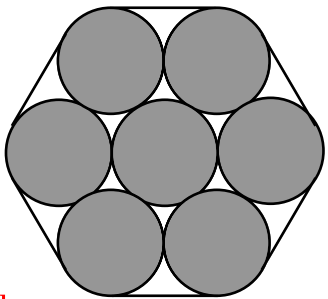
$$\text{and } \frac{C}{27}=\frac{15}{18}, \text{ so } C=\frac{45}{2}$$

$$\text{and } \frac{C}{X}=\frac{B}{15}, \text{ so X=15}$$

ANS: 15

3. The diagram depicts seven soft drink cans, seen from above, which are held tightly together by means of a ribbon. The circles represent the tops of the cans, and the other curve, which is clearly not a circle, represents the edge of the ribbon. The ends of the ribbon meet exactly; there is no overlap. Given that the cans all have diameter 6 cm, find the exact length of the ribbon.

.



Consider figures 1 and 2 below:

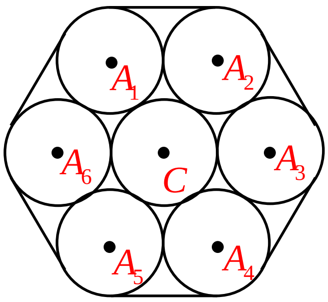


Figure 1

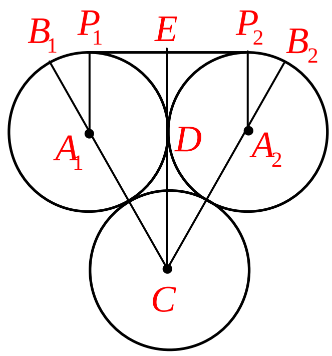


Figure 2

Let C be the centre of the centre can, and the number the cans around the outside 1, 2, 3, 4, 5, 6 in order. Let A_i be the centre of can i. For each i, draw a line CA_i and extend it to cut the ribbon at B_i. Then the ribbon is cut into 6 equal segments, and $\angle B_iCB_{i+1}=60^\circ$. Consider the segment between cans 1 and 2. Suppose the cans touch at D. then the line CD bisects angle $\angle B_1CB_2$, and A_1A_2 and CD are perpendicular. Let the point where the ribbon loses contact with the ith can be P_i, and let CD cut P_1P_2 at E, so CE and P_1P_2 are perpendicular.

The ribbon is tangent to can 1 at P_1, and so $\angle A_1P_1E=90^\circ$. Thus $P_1E=A_1D=3$ cm.

Also $\angle B_1A_1P_1 = \angle B_1CE = 30^\circ$. Therefore the length of the arc $B_1P_1 = \frac{1}{12} \times 6\pi = \frac{1}{2}\pi$.

So the length of ribbon from B_1 to E is $\frac{1}{2}\pi + 3$, and the length of ribbon from B_1 to B_2 is $2\left(\frac{1}{2}\pi + 3\right) = \pi + 6$. The total length of the ribbon is

$6(\pi + 6) = 6\pi + 36 = 54.84$.

ANS: 54.84 cm

4. *A*, *B*, *C* and *D* are four members of the football team. No two have the same weight. *A* is 8 kg heavier than *C*. *D* is 4 kg heavier than *B*. The sum of the weights of the heaviest and the lightest is 2 kg less than the sum of the weights of the other two people. If the sum of all their weights is 402 kg, what does *B* weigh?

Let *B* and *C* weight x kg and y kg respectively. Then *A* weights $y+8$ kg and *D* weights $x+4$ kg. Either *A* or *D* is the heaviest, and either *B* or *C* is the lightest. There are four possible cases to consider.

Case Ⅰ. *A is the heaviest and B is the lightest*. In this case

$$y+8+x = y+x+4-2$$

i.e. $x+y+8 = x+y+2$.

This is clearly incorrect.

Case Ⅱ. *A is the heaviest and C is the lightest*. In this case

$$y+8+y = x+x+4-2$$

i.e. $2y+8 = 2x+2$, or $y+3 = x$.

This implies that the weights of *A*, *B*, *C* and *D* are respectively $y+8$, $y+3$, y and $y+7$ kg. Their total weight is 402 kg, so we get $4y+18=402$, which gives $y=96$. The weights (in the same order) are 104 kg, 99 kg, 96 kg, and 103 kg. These figures are consistent with the information given in the question.

Case Ⅲ. *D is the heaviest and B is the lightest*. In this case

$$x+4+x = y+8+y-2$$

i.e. $2x+4 = 2y+6$, or $x = y+1$.

But if this is true, then *D*'s weight is $y+5$ kg, which less than *A*'s weight of $y+8$ kg, contradicting the assumption that *D* is the heaviest.

Case Ⅳ. *D is the heaviest and C is the lightest*. In this case

$$x+4+y = y+8+x-2$$

i.e. $x+y+4 = x+y+6$.

This is clearly incorrect.

Therefore Case Ⅱ gives the only possibility. It follws that *B*'s weight is 99 kg.

ANS: 99 kg

5. At a youth club one evening, one quarter of the members were playing pool, one sixth were playing table tennis, and five times the difference between these numbers were watching television. A further twelfth of the members were reading, leaving 7 members wandering around undecided what to do. How many

members were present that evening? (Please note: nobody was trying to do more than one thing at a time. For example, no one was playing pool and watching television.)

$5\times\left(\frac{1}{4}-\frac{1}{6}\right)=\frac{5}{12}$ of the members were watching television and so

$\frac{1}{4}+\frac{1}{6}+\frac{5}{12}+\frac{1}{12}=\frac{11}{12}$ of the members were occupied. That leaves $\frac{1}{12}$ of the members. We are told that there were 7 members wandering around undecided what to do. Assuming that all the references to members have been to members present, we see that 7 members constitute $\frac{1}{12}$ of the members present and so there were $12\times7=84$ members present that evening.

ANS: 84 members

6. In the sum shown, each digit 1, 2, 3, 4, 5, 6, 7, 8, 9 occurs just once.

$$\begin{array}{rccc} & & 3 & 9 \\ & & 4 & 5 \\ & & 7 & 8 \\ \hline \text{Total} & 1 & 6 & 2 \end{array}$$

There are many similar sums, in which three 2-digit numbers are added together to give a 3-digit number and each digit 1, 2, 3, 4, 5, 6, 7, 8, 9 occurs just once. What is the largest total (the 3-digit number)?

In the addition sum shown below, the different letters stand for different digits and there are no zeros.

$$\begin{array}{rccc} & & a & b \\ & & c & d \\ & & e & f \\ \hline \text{Total} & g & h & k \end{array}$$

The total is equal to $10(a+c+e)+(b+d+f)$, so the total is less than $10(7+8+9)+(4+5+6)=255$ and hence we can assume $g=2$ and $h\leqq5$ in order to find the largest total.

Note that if the sum of two of b, d and f is 10, then k is equal to the remainder one. This is clearly incorrect.

Case Ⅰ. $h=5$.

Thus (a, c, e) must be 7, 8, 9 and $b+d+f>10$. In this situation, the possible values of b, d, f and k are 1, 3, 4 and 6. So two of b, d and f is 4 and 6. This is clearly incorrect.

Case Ⅱ. $h=4$.

There are two possibilities.

(i) (a, c, e) is 7, 8, 9 and $b+d+f<10$. Then, the possible values of b, d, f and k are 1, 3, 5 and 6. So (b, d, f) is 1, 3, 5 and hence $k=1+3+5=9$. This contradicts with $k=6$.

(ii) (a, c, e) is 6, 8, 9 and $20>b+d+f>10$. Then, the possible values of b, d, f and k are 1, 3, 5 and 7. So (b, d, f) is 1, 5, 7 and hence

k=1+5+7－10=3. Thus we get a sum shown below:

		6	1
		8	5
		9	7
Total	2	4	3

So the largest total is 243.

ANS: 243

7. There is a duck-pond in the local park, with a one mile path right round it. One morning, grandma decided that she would have an hour's gentle exercise and walked round the pond at an average speed of 3 miles per hour. Her grandson Jerry started off at the same place and the same time, ran for an hour at an average speed of 8 miles per hour and went in the opposite direction to his grandmother. How many times did they meet after they had started and before they came to the end of their respective ordeals? (No including the final encounter.)

Grandma went round exactly 3 times and Jerry went round exactly 8 times. Since they went in opposite directions, Grandma met Jerry as often as she would have done if she had sat down and Jerry had gone round 11 times. The number of meetings, including the final encounter, would therefore have been 11 and so they met 10 times after they had started and before they came to the end of their respective ordeals.

ANS: 10

8. Catriona would like to become an Olympic sprinter. Her younger sister Morag would rather play football, but helps Catriona by racing against her. When they tried the 100 metre dash, Catriona cross the winning line when Morag was still 20 metres short of it. Catriona wanted something more challenging, so it was agreed that she would start 20 metres behind the normal starting line. They both ran at exactly the same speeds as in the first race. Where was Morag when Catriona crossed the winning line?

Morag ran 80 metres in the time that Catriona ran 100 metres so Morag runs at $\frac{4}{5}$ the speed of Catriona. Every race they run, Morgan will run only $\frac{4}{5}$ of the distance that Catriona runs in tha same time. While Catriona runs 120 metres, Morgan will run $\frac{4}{5}$ of 120 metres = 96 metres. Thus, Catriona crossed the winning line first and Morag was in the 4 metres short of the line.

ANS: the 4 metres short of the line

9. My bank card has a four digit number code that I need to punch in when I get money out of the ATM. To help me remember it I noted the following facts

(a) No two digits are the same.

(b) The fourth digit is the sum of the other three.

(c) The first digit is the sum of the middle two digits.

(d) If I reverse the number, the result as an exact multiple of 7

What is my number?

Let the number be *wxyz*, where *w*, *x*, *y* and *z* are digits. Form (b) we have $z=w+x+y$. Form (c) we have $w=x+y$. Hence $z=2w$. It follows that *z* is even and $w<5$. If $x=0$, then $w=y$, since $w=x+y$. This contradicts (a). Similarly, if $y=0$, then $w=x$, which again contradicts (a). Hence *x* and *y* are different digits, both greater than 0. Since $w=x+y$ it follows that the only possible values of *w* are 3 and 4.
If $w=3$, then $z=6$ and, since $w=x+y$, the digits *x* and *y* are either 1 and 2 respectively or 2 and 1 respectively. The number *wxyz* is then 3126 or 3216. Condition (d) requires the reversal *zyxw* to be an exact multiple of 7. However, neither 6213 or 6123 is an exact multiple of 7. We deduce that *w* cannot be 3.
Hence $w=4$ and $z=8$. Since $w=x+y$, the digit *x* and *y* are 1 and 3 respectively or 3 and 1 respectively (they cannot both be 2, since no two digits are the same). The number *wxyz* is then 4138 or 4318. When reversed, 8314 is not exactly divisible by 7, but 4318 reversed is 8134, which is exactly divisible by 7; in fact, $8134=7\times1162$. Therefore my number is 4318.

ANS: 4318

10.

	1	2	3	4	5	6

Six of seven positions are occupied by six frogs, which are numbered 1, 2, 3, 4, 5, 6 as shown. A frog can jump to an adjacent position if it is vacant, or leap over another frog to the next position but one if it is vacant, and can move backwards or forwards. What is the least possible number of moves required if the frogs are to occupy the six positions which were occupied at the start, but in the order 6 5 4 3 2 1 reading from left to right?

There are $\dfrac{6\times5}{2}=15$ pairs of frogs. The order of each pair must be reversed. A reversal requires a jump over another frog and each jump effects one reversal. Hence at least 15 jumps are needed. If the blank square is in an even position (its initial position or label 2, 4, 6) then at most three jump then moves the blank squares to one of the positions labeled 1, 3, 5 and at most two reversals can be achieved by 'leap-over' jumps.
Hence at least 15+6=21 jumps are needed. We can show that 21 jumps are in fact sufficient with an example:

- 1 2 3 4 5 6 → 2 1 - 3 4 5 6 → 2 1 4 3 - 5 6
→ 2 1 4 3 6 5 - → 2 1 4 3 6 - 5 → 2 1 4 - 6 3 5
→ 2 - 4 1 6 3 5 → - 2 4 1 6 3 5 → 4 2 - 1 6 3 5
→ 4 2 6 1 - 3 5 → 4 2 6 1 5 3 - → 4 2 6 1 5 - 3
→ 4 2 6 - 5 1 3 → 4 - 6 2 5 1 3 → - 4 6 2 5 1 3
→ 6 4 - 2 5 1 3 → 6 4 5 2 - 1 3 → 6 4 5 2 3 1 -
→ 6 4 5 2 3 - 1 → 6 4 5 - 3 2 1 → 6 - 5 4 3 2 1
→ - 6 5 4 3 2 1

ANS: 21

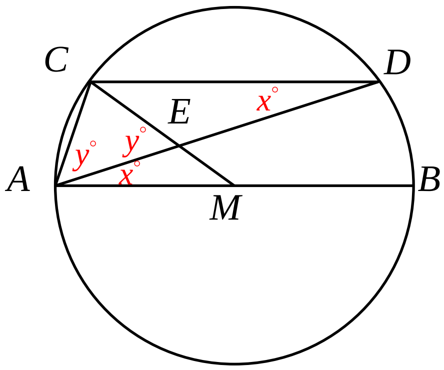
11. As I am sure you know, a date can be represented by writing down three positive integers. For example, the seventeenth of June 1995 can be represented by 17.6.95; it is the 17th day of the 6th month in the 95th year of the 20th century. Define the "date sum" of any date to be the sum of the corresponding integers. Thus the date sum of 17 June 1995 is 17+6+95=118. Now let n be the sum of the date sums of all the days from 1st January 2001 to 31st December 2007 inclusive. Find the value of n.

We shall refer to the three components as the day number, the month number and the year number. To find n, we can first add all the day numbers together to obtain a number d, then all the month numbers to obtain a number m and finally all the year number to obtain a number y. Then $n=d+m+y$.

There are one leap year (2004) and 7×12=84 months between 2001 and 2007, so there are 7×7=49 of these months have 31 days, 7×4=28 of these months have 30 days, 1 of these months has 29 days and 6 of these months has 28 days. Hence
$d=(1+2+3+\ldots+28)\times 84+29\times 78+30\times 77+31\times 49=40195$,
$m=(1+3+5+7+8+10+12)\times 31\times 7+(4+6+9+11)\times 30\times 7+2\times 29\times 1+2\times 28\times 6=16676$,
and $y=(1+2+3+5+6+7)\times 365+4\times 366=10224$.
So $n=40195+16676+10224=67095$.

ANS: 67095

12. AB is a diameter of a circle with centre M. CD is a chord of this circle which is parallel to AB and C is its extremity nearer to A. MC meets AD at a point E such that $AC=EC$. Find the size of the angle CAM.



Let $\angle CAE= y°$ and $\angle DAM= x°$. Then

$\angle CDA=\angle DAM= x°$ (alternate angles)

$\angle CMA=2\angle DAM=2x°$ (angle at centre)

$\angle CEA=\angle CAE= y°$ ($\triangle ACE$ is isosceles)

But $\angle CEA+\angle AEM= 180°$ and $\angle EAM+\angle AME+\angle AEM = 180°$ means that $\angle CEA =\angle EAM+\angle AME$, i.e. $y=x+2x=3x$.

But $AM=MC$ since they are radii of the circle, thus $\triangle AMC$ is isosceles and $\angle ACM=\angle CAM= x°+ y°$. So from the sum of the angles of $\triangle AEC$

$$x+y+y+y=180$$
$$10x=180$$
$$x=18$$

and $y=54$. Thus $\angle CAM= 72°$

ANS: 72°