INTERNATIONAL MATHEMATICS AND SCIENCE OLYMPIAD FOR PRIMARY SCHOOLS (IMSO) 2008

Mathematics Contest (Second Round) in Taiwan, Essay Problems

Name:_______ School:_______ Grade:______ ID number:________

Answer the following 10 questions, and show your detailed solution in the space provided after each question. Each question is worth 4 points.
Time limit: 60 minutes.

1. Find which is the greatest, 2^{9972}, 65^{1662} or 33^{1995}.

   Since $33 > 32 = \frac{1}{2} \times 64 = \frac{1}{2} \times 2^6$, we have

   $$\frac{33^{1995}}{2^{9972}} = \frac{33^{1995}}{64^{1662}} = 33^{333} \times \left(\frac{33}{64}\right)^{1662} > 32^{333} \times \left(\frac{1}{2}\right)^{1662} = 2^{5\times 333} \times \left(\frac{1}{2}\right)^{1662} = 2^3 > 1.$$

   Therefore, $33^{1995} > 2^{9972}$.

   Since $33 > \frac{1}{2} \times 65$, we have $\frac{33^{1995}}{65^{1662}} = 33^{333} \times \left(\frac{33}{65}\right)^{1662} > 32^{333} \times \left(\frac{1}{2}\right)^{1662} = 2^3 > 1$.

   Therefore, $33^{1995} > 65^{1662}$. So 33^{1995} is the greatest one.

   ANS: 33^{1995}

2. Sarah and her young brother Stephen are earning pocket-money by mowing lawns for their neighbors. Sarah uses the lawn mower and hopes to make \$5 an hour for herself. Stephen helps by taking away the cuttings and tidying up. They agree that he will receive 20% of the total payment made for doing the work. It will take them four hours to mow the lawn in the garden of a large house nearby. How much will the owner pay if Sarah is to earn for herself exactly \$5 an hour?

   Sarah wants \$5 per hour, so altogether she wants to earn \$5×4=\$20 for herself. Stephen receives 20 % of the total payment, and so Sarah receives 80%. Therefore she receives four times as much as Stephen does. It follows that, if she is to receive \$20, Stephen will receive \$5 and so the man must pay \$20+\$5=\$25.

   ANS: \$25

3. John cycles to Jane's house and back along a flat straight road, giving him a round trip of 32 km. During the whole of the time both the speed and direction of the wind are constant. On the way out, the wind is directly behind John and so may be added to his normal bicycle speed to give his speed for this part of the journey, but on the way back the wind opposes him and its speed must be subtracted from his bicycle speed. When John arrives back home he finds that his total travelling time has been 2 hours and that the homeward part of his journey has taken twice as long as the outward part. Find the speed of the wind, in km/hour.

   Total time = time out + time back = 3×time out, since time back is twice time out. But total time = 2 hours, so time out = 2/3 hours and time back = 4/3 hours.

OUT:

distance = 16 km, time = 2/3 hours, speed = normal speed + wind speed.

BACK:

distance = 16 km, time = 4/3 hours, speed = normal speed − wind speed.

SPEED=DISTANCE÷TIME and so

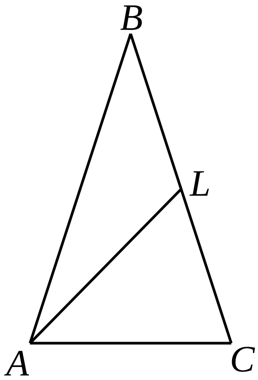
normal speed + wind speed = 16÷(2/3) = 24 km/hour

normal speed − wind speed = 16÷(4/3) = 12 km/hour

Difference between speeds = 2×wind speed = 12 km/hour and hence wind speed is 6 km/hour.

ANS: 6 km/hour

4. In a triangle ABC, $AB=BC$. L is the point on BC such that AL bisects $\angle BAC$. If $AL=AC$, find the size of the angle of the triangle ABC.



Let $\angle LAC=x°$. Then $\angle BAC=2x°$ as AL bisects $\angle BAC$.

Since $AB=AC$, $\angle BAC=\angle BCA$ (isosceles triangle).

Hence $\angle BCA=2x°$.

On the other hand, $\angle ALC=\angle ACL$ as $AL=AC$ (isosceles triangle).

Hence $\angle ALC=2x°$.

Since the angle sum of a triangle is 180°, $\angle ALC+\angle ACL+\angle LAC=180°$

i.e. $5x°=180°$.

Hence $x=36$. Therefore, $\angle BAC=72°$ and $\angle ABC=180°-72°-72°=36°$.

ANS: $\angle ABC=36°$

5. In the first stage of a baseball competition, the teams are divided into grounds of four. Each team in a group plays the three other teams once each, and the results are drawn up in a “group table”, which records the number of wins, draws and defeats for each team, and the number of runs scored and against. Three points are allocated for a win, one for a draw and none for a defeat. Here is part of a group table

Team	Runs		Points
	Scored	Against	
A	2	2	4
B	4	4	6
C	2	1	4
D	1	2	2

Find the results and the scores in all the matches played in the group.

From the total numbers of points:

A had 1 win, 1 draw and 1 defeat
B had 2 wins, no draw and 1 defeat
C had 1 win, 1 draw and 1 defeat
D had no wins, 2 draw and 1 defeat

C had a defeat, but only 1 run “against”. Therefore the score in the defeat was 0-1, the score in the draw was 0-0 and the score in the win was 2-0. *B* had 4 runs “against”. Since *A*’s total of runs “scored” was 2 and *D*’s total was 1, *C* must have scored at least 1 against *B*. But *C* scored in only one game, which they won 2-0. Hence the score in *B* vs *C* was 0-2. *D* had no wins, so the score in *A* vs *C* was 1-0 and in *C* vs *D* was 0-0. Summarizing *C*’s matches: *C* vs *A* 0-1, *C* vs *B* 2-0, *C* vs *D* 0-0.

Since *B*’s score against *C* was 0-2, all *B*’s 4 runs were scored against *A* and *D*. Since *A* and *D* each had totals of 2 runs “against”, *B* scored 2 against each. *B* had a total of 4 “against”; 2 were scored by *C*. This leaves a total of 2 scored by *A* and D against *B* and since *B* defeated both *A* and *D*, the score was 2-1 in each case. Now we have the results and scores of all *B*’s matches: *B* vs *A* 2-1, *B* vs *C* 0-2, *B* vs *D* 2-1.

There is only one match left: *A* vs *D*, which was a draw 0-0, since all *A*’s runs scored and against have been accounted for.

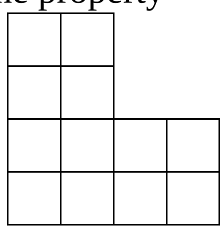
Thus the results and scores are as indicated by the following table:

	A	***B***	***C***	***D***
A		1-2	1-0	0-0
B	2-1		0-2	2-1
C	0-1	2-0		0-0
D	0-0	1-2	0-0	

6. The number 132 has three digits, no two of which are equal. It has the property of being equal to the sum of all the different 2 digit numbers made up from its three digits, viz.

$$132=13+12+21+23+31+32$$

Find all other such 3 digit numbers.



Let $\overline{abc}$ be a such 3 digit number, with *a*, *b* and *c* are different. Thus we have

$$\overline{abc}=\overline{ab}+\overline{ba}+\overline{bc}+\overline{cb}+\overline{ca}+\overline{ac}$$

i.e. $100a+10b+c = 10a+b+10b+a+10b+c+10c+b+10c+a+10a+c=22a+22b+22c$ or $26a-4b = 7c$.

Since $26a-4b$ is even, c is also an even number. Assume $c=2k$, where $k=1, 2, 3, 4$. Then $13a-2b = 7k$ or $13a = 7k + 2b$. Since $7k + 2b<28+18=46$, $a=1, 2$ or 3. Note that $7k<13a$ and the values of a and k are both odd or even since $13a-7k = 2b$ is even.

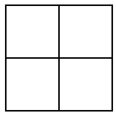
(i) $a=1$. Then $k=1$ and hence $b=3$. So $\overline{abc}=132$.

(ii) $a=2$. Then $k=2$ and hence $b=6$. So $\overline{abc}=264$.

(iii) a=3. Then k=3 and hence b=9. So $\overline{abc} = 396$.
Hence the numbers which we want to find are 264 and 396.

ANS: 264 and 396

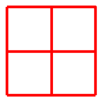
7. Color the twelve small squares in the diagram, using three colors altogether, in such a way that
   (a) No two squares which have an edge in common have the same color.
   (b) Each of the three colors is used in exactly four of the small squares.
   (c) In any four small squares forming a block as part of the diagram, all three colors are used.



(Note: You must not mix the colors or use different colors in the same small square. Using B, G, R as three colors)

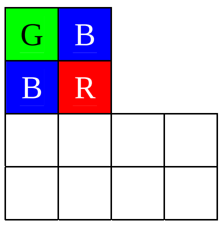
There are many possible colorings that work here. For all four marks you need either to explain a systematic way of doing the coloring, or to demonstrate clearly how your coloring obeys the rules. Here is an example description of a 'systematic' method:

Developing a systematic method:

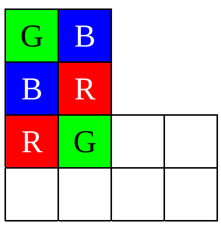
(i) In any block of 4 squares : 

- Rule (c) means that only one color is repeated;
- Rule (a) means that the repeated color is on diagonally opposite squares.
- Since the L-shape can be separated into 3 blocks of 4, each of which must contain each color, Rule (b) means that the repeated color cannot be the same in any two of these three blocks (or we would have more than 4 squares of that color in total).

(ii) Start by coloring the first block of 4 in this way:



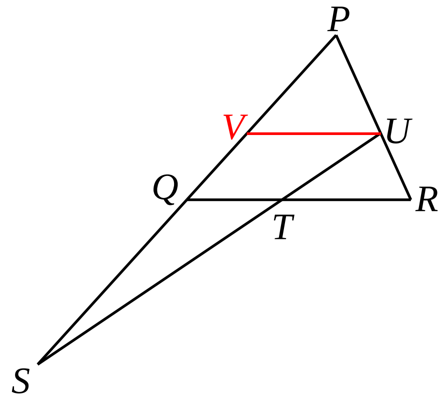
then color the next two squares down so that the new, overlapping, block of 4 works;



continue in this way until the whole shape is completed.

G	B		
B	R		
R	G	B	R
G	B	R	G

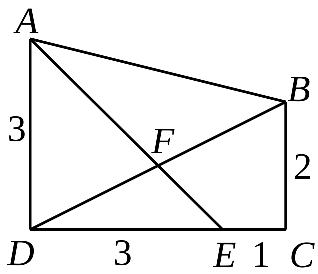
8. PQR is a triangle. PQ is extended to S so that $PQ=QS$ and U is a point on PR such that $PU:UR=3:2$. T is the point of intersection of the lines QR and SU. Find $QT:QR$.

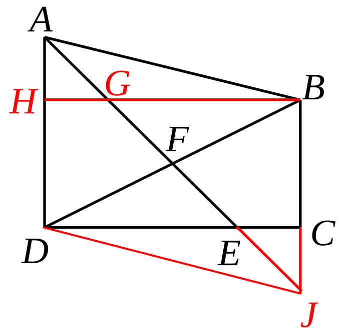


Construct $VU//QR$ with V on PQ. Then
$PV:VQ=PU:UR=3:2$ ($\triangle PVU$ and $\triangle PQR$ are similar)
$PQ:VQ=5:2$
$QS:VQ=5:2$ ($PQ=QS$)
$QS:VS=5:7$
$QT:VU=QS:VS=5:7$ ($\triangle QTS$ and $\triangle VUS$ are similar)
Also $VU:QR=PU:PR=3:5$ ($\triangle PVU$ and $\triangle PQR$ are similar)
Hence $QT:QR=3:7$

ANS: 3:7

9. In the quadrilateral $ABCD$ the sides AD and BC are parallel and AD is perpendicular to DC. The lengths of AD and BC are 3 and 2, respectively. E is the point on DC between D and C such that DE has length 3 and EC has length 1, as shown. The lines AE and BD meet at F. Find the exact area of triangle ABF.





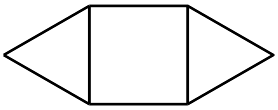
Extend AE to meet BC at J. Draw HB parallel to DC, intersecting AE at G. $\angle DAE=\angle DEA=45°$. Thus both $\angle HGA$ and $\angle CEJ$ are 45°.
Also $HG=AH=AD-HD=AD-BC=1$ unit; similarly $CJ=1$ unit. $ABJD$ is a parallelogram, with area $3\times4=12$ square units, and F is the point of intersection of its diagonals, the midpoint of each.

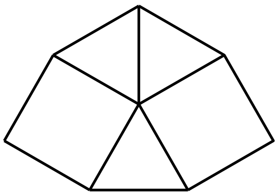
Thus $\triangle ABF=\frac{1}{2}\triangle ADB=\frac{1}{4}$(area of $ABJD$)=3 square units.

ANS: 3 square units

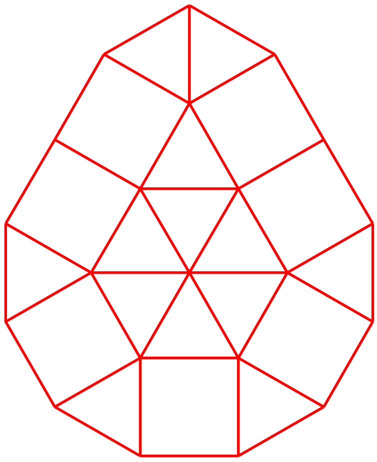
10. Equilateral triangles and squares, each with sides of unit length, can be used to construct convex polygons. For example, two triangles and a square can be put

together to form a hexagon, and three triangles and two squares to form a 7 side polygon, as shown in the diagrams. (The region enclosed by the polygon must be covered exactly by the triangles and squares used in the construction.) How the process can be used to construct a convex polygon with 11 sides?





The angles at the vertices of a convex polygon are all less than 180°. Since the angles of an equilateral triangle are all 60°, the possible angles in the figure are 60°, 90°, 120° or 150°. It follows that each external angles is at least 30°. Therefore, since the sum of the external angles is 360°, the number of sides is at most 360°÷30°=12. The diagrams below show how 11-sided polygons can be constructed using the process.



11-sided