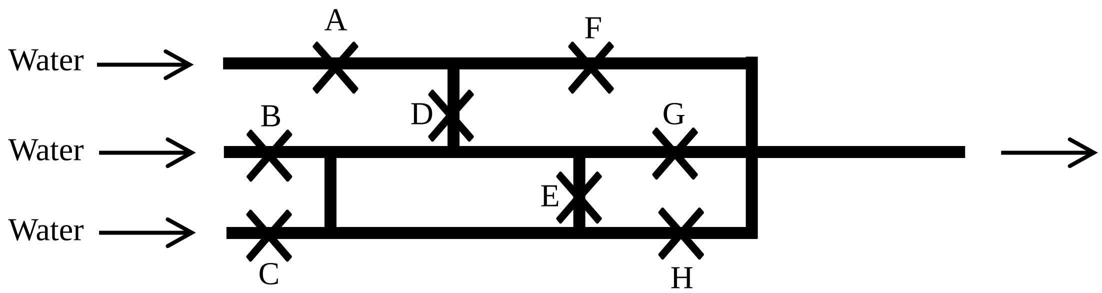
INTERNATIONAL MATHEMATICS AND SCIENCE OLYMPIAD FOR PRIMARY SCHOOLS (IMSO) 2008

Mathematics Contest in Taiwan, Exploration Problems

Name:_______ School:_______ Grade:______ ID number:________

Answer the following 5 questions. Write down your answer in the space provided after each question. Each question is worth 8 points. Time limit: 60 minutes.

1. The diagram shows a network of water pipes, with eight taps, each shown by a symbol ×, which are at positions A, B, C, D, E, F, G and H. Water enters along three of the pipes at the left, as shown by the symbol →, and is prevented from making any progress past a tap which is in the "off" position. Taps are either "on" or "off". How many ways of taps in the "on" position will result in water coming out of the pipe on the extreme right, assuming that no more than three taps can be in the "on" position at the same time?



The possible combinations are

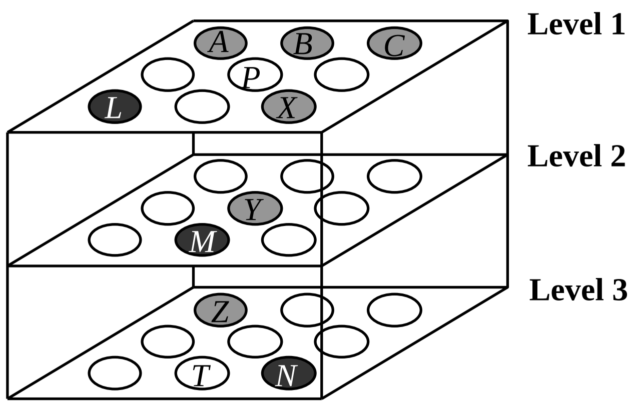
(1) with exactly two taps on: AF, BG, BH, CG, CH,

(2) with exactly three taps on: ADG, ADH, BDF, CDF, ABF, ABG, ABH, ACF, ACG, ACH, ADF, AEF, AFG, AFH, BCG, BDG, BEG, BFG, BGH, BCH, BDH, BEH, BFH, CDH, CEH, CFH, CGH, CDG, CEG, CFG.

So there are totally 35 ways.

ANS: 35

2. The diagram shows a 3×3×3 frame with 27 holes in which marbles can be placed. *A*-*B*-*C*, *L*-*M*-*N* and *X*-*Y*-*Z* are some possible lines of three marbles.
   (a) From *A*, how many different lines of three marbles are possible?
   (b) What is the total number of different possible lines of three marbles in this frame?

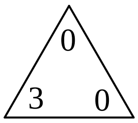


(a) There are three 'short' lines of the type *A*-*B*-*C*, three 'medium' lines of the type *L*-*M*-*N* (we include the line *A*-*P*-*X* as a 'medium' line) and one 'long' line of the type *X*-*Y*-*Z*. So 7 different lines are possible from *A*.

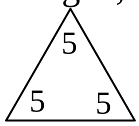
(b) There are 27 ‘short’ lines (such as type *A*-*B*-*C*), 18 ‘medium’ lines (such as type *A*-*P*-*X*) and 4 ‘long’ lines (such as type *A*-*Y*-*N*) for a total of 49 different lines.

ANS: (a) 7 (b) 49

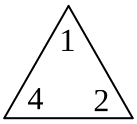
3. Triominoes is played with a set of tiles, all sides of which are equal in length. Each vertex of every tiles is marked with one of the digits from 0 to 5. Different vertices on the same tile can have the same digits, so that



and



are both in the set. All possible tiles with duplicate or triplicate digits, like those illustrated, are included. How many of all possible tiles are there? (The tiles



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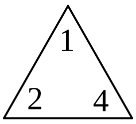
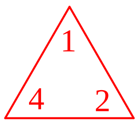


are different tiles.)

There are 6 tiles with triplicate digits. For duplicate tiles there are 6 choices of the repeated digit. For any one of the these there are 5 choices for the third digit and so there are 6×5=30 triominoes of type altogether. The different possibilities for three different digits are

(0, 1, 2), (0, 1, 3), (0, 1, 4), (0, 1, 5), (0, 2, 3), (0, 2, 4), (0, 2, 5), (0, 3, 4), (0, 3, 5), (0, 4, 5), (1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 4), (1, 3, 5), (1, 4, 5), (2, 3, 4), (2, 3, 5), (2, 4, 5), (3, 4, 5).

Each of 3 digits can give rise to two distinguishable triominoes. For example, we can distinguish between



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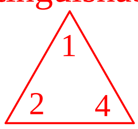

since the order 1, 2, 4 is “clockwise” in the first case and “anti-clockwise” in the second. It follows that in the set there could have been 40 different tiles with digits all different. Hence there are totally 40+30+6=76 possible tiles.

ANS: 76

4. At the end of the season, the total prize money in a football tipping competition is to be shared among the 5 place getters according to the following rules:
   (i) All prizes are different and each is a positive whole number of dollars;
   (ii) First prize is the sum of second and third prized;
   (iii) Second prize is the sum of fourth and fifth prizes;
   (iv) The higher the placing, the larger the prize. (Assume that there are no ties among the place getters.)

(1) Find the smallest amount of total prize money possible for which these rules may apply.

(2) If the total prize money available is \$72, find all possible ways in which the \$72 may be distributed among the place getters.

(1) If the fifth prize is \$1 and the fourth prize is \$2, the second prize is \$3. This leaves no possible value for the third prize if it is to be different from the second and fourth prizes. The same argument applies for any sized fourth prize of \$1 for fifth. Therefore the fifth prize must be worth more than \$1.

If the fifth prize is \$2 and the fourth prize is \$3, then the second prize is \$5 and the third prize must be \$4. Then the first prize is \$9. The set of values for the 5 prizes is the least possible. The total prize money is \$(2+3+4+5+9)=\$23.

(2) We can tabulate until a third prize works.

1st (=2nd+3rd)	2nd (=4th+5th)	3rd	Possible?
24	24	0	No (0 not possible)
25	22	3	Yes
26	20	6	Yes
27	18	9	Yes
28	16	12	Yes
29	14	15	No (2nd<3rd)

If the first prize is \$24, then the second and third prizes combined are \$24. This leaves \$24 for the fourth and fifth prizes, which must add to the second prize. Then the second prize is \$24 and the third prize must be \$0. But each prize must be a positive whole number of dollars. So this is not possible.
If the first prize is less than \$24 then the sum of the second and third prizes is less than \$24. The sum of the fourth and fifth prizes is larger than \$24 and therefore the second prize is larger than \$24. This means the third prize is a negative number. So this is not possible.
If the first prize is \$29, then the second and third prizes combined are \$29. This leaves \$14 for the fourth and fifth prizes, which must add to the second prize. Then the second prize is \$14 and the third prize must be \$15. But the second prize must greater than the third prize, and therefore more than half of \$29. So this is not possible.
If the first prize is larger than \$29 then the sum of the second and third prizes is larger than \$29. The sum of the fourth and fifth prizes is less than \$14 and therefore the second prize is less than \$14. But the third prize is greater than \$15 then and this is not within the rules.
Hence we must try the first prize of \$28, \$27, \$26 and \$25.

1st	2nd	3rd	4th	5th
28	16	12	11	5
28	16	12	10	6
28	16	12	9	7
28	16	12	8	4th=5th
27	18	9	8	4th <5th
26	20	6	4th or 5th >3rd	
25	22	3	4th or 5th >3rd	

So there are three ways to allocate the prizes:

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
28	16	12	11	5
28	16	12	10	6
28	16	12	9	7

5. An n-dragon is a set of n consecutive positive integers. The first two-thirds of them is called the *tail*, the remaining one-third is called the *head* and the sum of the numbers in the tail is equal to the sum of the numbers in the head. For example, the 9 consecutive integers 2, 3, 4, 5, 6, 7, 8, 9 and 10 form a 9-dragon.
   - Its tail is 2, 3, 4, 5, 6 and 7, i.e. six numbers with sum 27.
   - Its head is 8, 9 and 10, i.e. three numbers with sum 27.

(1) Find a 21-dragon.
(2) Find the sum of the tail of a 99999-dragon.

(1) The 3-dragon 1, 2, 3 starts with 1 and the 9-dragon 2, 3, 4, …, 10 starts with 2. We expect the next dragon to be a 15-dragon and that it starts with 3. This checks since

$$3+4+5+6+7+8+9+10+11+12=75$$
$$\text{and } 13+14+15+16+17=75.$$

The next dragon will be the 21-dragon starting with 4. This checks since

$$4+5+6+7+8+9+10+11+12+13+14+15+16+17=147$$
$$\text{and } 18+19+20+21+22+23+24=147.$$

So 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23 and 24 form a 21-dragon.

(2) The first number in an n-dragon for small values of n is

n	3	9	15	21
First number	1	2	3	4

These seem to fit the rule in which the first number of an n-dragon is $\frac{n+3}{6}$.

So we suspect that for a 99999-dragon, the first number might be $\frac{99999+3}{6}=16667$. The tail will contain 66666 numbers, so the last one would be 16668+66665=83332. We can find double the sum of the numbers in the tail by adding up its 66666 numbers like this
(16667+83332)+(16668+83331)+…+(83332+16667)=66666×99999,
since all the bracketed terms add to 99999. So the sum of the tail is half this: 33333×99999=3333266667. Thus head sum can be calculated similarly. The first number in the head is 83333 and the last is 83333+33332=116665, so double the sum of the numbers in the head is
(83333+116665)+(83334+116664)+…+(116665+83333)=333333×199998,
since all the bracketed terms add to 199998. So the head sum is half of this: 33333×99999=3333266667. So the head sum is the same as the tail sum so there is a 99999-dragon with tail sum 3333266667.