INTERNATIONL MATHEMATICS AND SCIENCE OLYMPIAD FOR PRIMARY SCHOOLS (IMSO) 2006

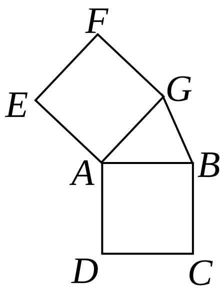
Mathematics Contest (Second Round) in Taiwan, Essay Problems

Name:________ School:_______ Grade:______ ID number:_________

Answer the following 10 questions, and show your detailed solution in the space provided after each question. Each question is worth 4 points.
Time limit: 60 minutes.

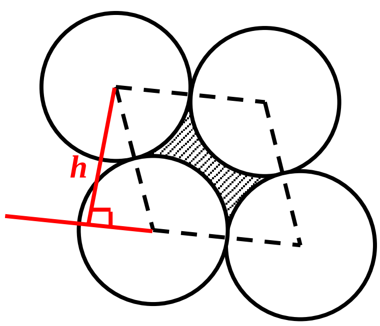
1. John challenged Tom to a general knowledge contest. For every answer Tom got right, John gave him 7 \$ and for every answer Tom got wrong, he had to give John 3 \$. John asked 50 questions and at the end of the competition, when everything was added up, neither owned the other any money! How many questions did Tom answer correctly?
   If Tom got 3 right answers, then he won $3\times7=21$ \$. If Tom got 7 wrong answers, then he lose $7\times3=21$ \$. So if there is an average of 3 right answers in 10 questions Tom got, then neither owned the other any money.
   Since $50=10\times5$, Tom answered $5\times3=15$ questions correctly.

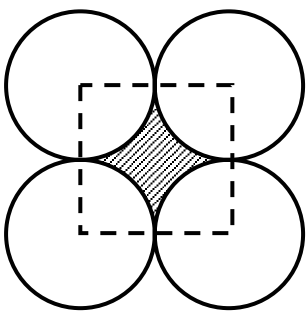
2. The diagram shows two squares, *ABCD* and *AEFG*, which are equal in size. Giving full reasons for each of your statements, show that $\angle DAE=2\times\angle ABG$.



   Since *ABCD* and *AEFG* are equal squares, $AG=AB$ and hence $\angle ABG=\angle AGB$. Observe that
   $\angle DAE+\angle EAG+\angle GAB+\angle BAD=360°$.
   Since $\angle EAG=\angle BAD=90°$,
   $\angle DAE+\angle EAG+\angle GAB+\angle BAD=\angle DAE+\angle GAB+180°$.
   So $\angle DAE+\angle GAB=180°=\angle GAB+\angle ABG+\angle AGB$,
   i.e. $\angle DAE=\angle ABG+\angle AGB=2\times\angle ABG$.

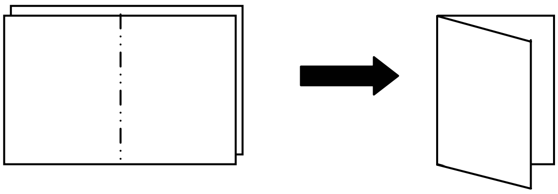
3. The diameter of a coin is 1 cm. If four such identical coins are placed on a table that each rim of them touches the other two, find the maximum possible area that is enclosed inside. ($\pi=3.14$)





As the above figures, the area that is enclosed inside is the shaded area and equal to the area of a parallelogram minus the area of a circle. Since the area of the parallelogram is equal to $(1+1)\times \boldsymbol{h} \leqq (1+1)\times(1+1)$. So the area of a square is greater than parallelograms with the same perimeter and the maximum possible area is equal to the area of a square minus the area of a circle. So the answer is $(1+1)^2-3.14\times1^2=4-3.14=0.86$ (cm^2).

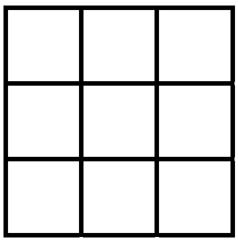
4. An A5 booklet is constructed by stapling 20 sheets of A4 paper together down the middle and then folding.



Starting with the front page, the pages are numbered consecutively from 1 to 80. A certain page is numbered 30. If the book is then unstapled, which other three numbers will appear on this same sheet of paper?

Observe that the sum of the numbers on the same side is 81 and the four numbers on the same paper must be of the form *n*, *n*+1, *m* and *m*+1, where *m* and *n* are odd numbers. So the other three numbers are 29, 51 and 52.

5. In how many ways can an ☆ be placed in the cells of the grid shown so that each row and each column contains exactly two cells with an ☆ ?



When you fix the situation of the first row, there are 2 possible ways to make each row and each column contains exactly two cells with an ☆:

	☆	☆
☆	☆	
☆		☆

	☆	☆
☆		☆
☆	☆	

Since there are 3 choices for the first row, there are 3×2=6 ways to make each row and each column contains exactly two cells with an ☆.

6. Amy drives her car at constant speed. At 1 pm, 2 pm and 3 pm, she notices her distance from home. When she starts her journey at 1pm, her distance from home is a two digit number of kilometres, after an hour it is given by the same two digits in reverse order and after a second hour the distance is given by the original two digits separated by a zero. Calculate Amy's speed.

Let the distance from home be $\overline{ab}$ at 1 pm, $\overline{ba}$ at 2 pm and $\overline{a0b}$ at 3 pm, where $\overline{ab}$ means $10a+b$, $\overline{ba}$ means $10b+a$ and $\overline{a0b}$ means $100a+b$. for $0<a, b<10$. Assume Amy drives her car at $\boldsymbol{v}$ km/hour, then

$$2\boldsymbol{v}=\overline{a0b}-\overline{ab}=90a$$

$$\boldsymbol{v}=\overline{ba}-\overline{ab}<90$$

$$\therefore\ \boldsymbol{v}=45a<90$$

Since $0<a<10$, we have $a=1$, $\boldsymbol{v}=45$ km/hour.

7. Eric has an odd way of counting his gold coins. Firstly, he splits them up into two piles with numbers of coins in the ratio of 1:2. He then splits the smaller pile into two piles in the ratio 1:3. Then he splits the new smaller pile into two piles in the ratio 1:4 and then 1:5, 1:6. Eventually, having carried on some hours, he splits the smaller pile in the ratio 1:7 and finds that the new small pile only contains one coin. How many gold coins does Eric have in total?

   Eric has 7 piles of gold coins finally. Assume a, b, c, d, e, f and g are the numbers of each pile, where $a \leqq b \leqq c \leqq d \leqq e \leqq f \leqq g$. Thus we have:

$$g=2(a+b+c+d+e+f)$$
$$f=3(a+b+c+d+e)$$
$$e=4(a+b+c+d)$$
$$d=5(a+b+c)$$
$$c=6(a+b)$$
$$b=7a$$

So all gold coins are $a+b+c+d+e+f+g=3(a+b+c+d+e+f)=3\times4(a+b+c+d+e)$ $=3\times4\times5(a+b+c+d)=3\times4\times5\times6(a+b+c)=3\times4\times5\times6\times7(a+b)=3\times4\times5\times6\times7\times8a=20160a$. Because the smallest pile only contains one coin, $a=1$ and hence Eric have 20160 gold coins.

8. A radio presenter invites listeners to "crack the code" to win a prize. The code is a 4-digit number, which may start with a 0. All the digits are different and are arranged in ascending order, so 2457 is a possible code, but 1973 and 2448 are also not. A caller guesses 0389 and is told that she has no number correct; a second then guesses 1456, and is told that she has three numbers correct and two of these are in the correct place. After this, the presenter giving a clue that this code is an even number. Is it now possible to "crack the code", i.e. given these two guesses and the presenter's responses, to give exactly one correct code fitting all the conditions. If not, why not?

   Since the first caller has no number correct, the correct numbers are four of 1, 2, 4, 5, 6 and 7. The second guess is 1456 and one of the four numbers is wrong, so we have the following situations:

(1) 1 is wrong number:

The possible answers are 2456 and 4567. Since only two numbers are on the right place, the two 4-digit numbers are not the answer.

(2) 4 is wrong number:

The possible answers are 1256 and 1567. Since only two numbers are on the right place, the two 4-digit numbers are not the answer.

(3) 5 is wrong number:

The possible answers are 1246 and 1467. Since 1467 is odd number, only 1246 satisfy the conditions.

(4) 6 is wrong number:

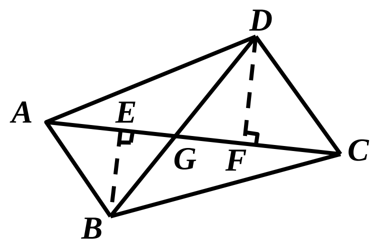
The possible answers are 1245 and 1457. Since only two numbers are on the right place, the two 4-digit numbers are not the answer.

So we can crack the code now. The code is 1246.

9. Is it possible to write down more than 50 different 2-digit numbers in such a way that there aren't two numbers with the sum equal to 100? If it is possible, please construct an example. If not, explain why.

Yes, it is. The following numbers are one possibility:
50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,99.
In fact, there are 40 situations to write 100 as a sum of two different two-digit numbers. When we write down all of them as number pairs, they are of the form $(10+k, 90-k)$, where k is an integer between 0 and 39. If you pick one number from each pair and get together with 50, 91, 92, 93, 94, 95, 96, 97, 98 and 99, then you get one solution.

10. The lengths of both diagonals of a quadrilateral are no longer than 1. What is the largest possible area of the quadrilateral?



As figure, the area of the quadrilateral is equal to the sum of $\triangle ACD$ and $\triangle ABC$. So the area of the quadrilateral = $\frac{1}{2}AC(BE+DF)$. Because $BE+DF \leqq BG+DG=BD$, hence the largest possible area of the quadrilateral occurs when $BD \perp AC$ and the area of the quadrilateral = $\frac{1}{2}AC \times BD$.

Since $AC \leqq 1$ and $BD \leqq 1$, the largest possible area of the quadrilateral is $\frac{1}{2}$.