INTERNATIONAL MATHEMATICS AND SCIENCE OLYMPIAD FOR PRIMARY SCHOOLS (IMSO) 2007

Mathematics Contest (Second Round) in Taiwan, Essay Problems

Name:________ School:________ Grade:_______ ID number:__________

Answer the following 10 questions, and show your detailed solution in the space provided after each question. Each question is worth 4 points.
Time limit: 60 minutes.

1. The 400-digit number 12345678901234567890···890 is given.
Step 1: Cross out all the digits in odd-numbered places.
Step 2: Cross out all the digits in odd-numbered places of the remaining digits.
…
Continue until no digits remain. What is the last digit to be crossed out?
Observe the following facts.
After step 1, the remaining digits are in even-numbered places of the origin number.
After step 2, the remaining digits are in numbered as the multiple of $4=2^2$ places of the origin number.
After step 3, the remaining digits are in numbered as the multiple of $8=2^3$ places of the origin number.
…
And so on.
Since the number is a 400-digit number , $2^8=256<400<512=2^9$, the final digit will be cross out at step 9. So the last digit is in numbered 256, i.e. the last digit is 6.

2. A street of houses numbered from 1 to 302 inclusive is to be numbered with new brass numerals. How many of the digits "2" would be needed to complete the job?
There are 31 "2" on the unit digit, 30 "2" on the tens' digit and 100 "2" on the hundreds' digit, so there totally 31+30+100=161 "2" are needed.

3. Articles X, Y and Z are for sale. Article X can be bought at the rate of eight for \$1. Article Y costs \$1 each and Article Z costs \$10 each. You buy a selection of all three types and find that you have to purchased exactly 100 articles at a cost of \$100. How many articles of type Y did you buy?
Assume you bought x X's, y Y's and z Z's. Then $x+y+z=100$ and $\frac{1}{8}x+y+10z=100$.
$$\begin{cases} x+y+z=100 \\ x+8y+80z=800 \end{cases} \Rightarrow 7y+79z=700 \Rightarrow y=100-\frac{79}{7}z$$
Since x, y and z are positive integers, z must be 7 and hence $y=21$ and $x=72$. So you bought 21 articles of type Y.

4. Two candles have different lengths and thicknesses. The longer one can burn for 7 hours and the shorter one for 10 hours. After 3 hours' burning, both candles have the same length. What was the shorter candle's length divided by the longer candle's length?

Let x and y be the length of the shorter candle and the longer candle, respectively. After 3 hours' burning, the longer candle can still burn 4 hours and the shorter candle can burn 7 hours. So $\frac{7}{10}x=\frac{4}{7}y$, i.e. $\frac{x}{y}=\frac{40}{49}$.

5. John had a summer job on a farm. He had four bags of potatoes to weight but each bag weighed more than 60 kg and less than 100 kg , the scale only weighed in excess of 100 kg. He solved the problem by weighing the bags two at a time. He found the weightings to be 124, 132, 134, 138, 140 and 148. What was the weight of the lightest bag, in kilograms?

Let a, b, c and d be the weights of the four bags and $a<b<c<d$. Thus we have:

$$a+b<a+c<b+c<b+d<c+d \text{ and } a+b<a+c<a+d<b+d<c+d.$$

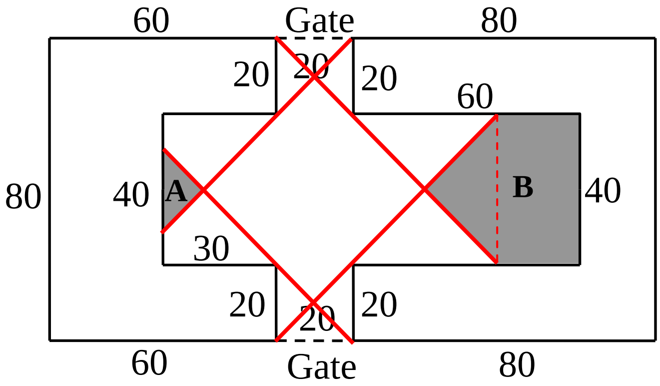
So $a+b=124$, $a+c=132$, $b+d=140$ and $c+d=148$.

If $b+c=138$, then $b-a=6$ and hence $a=59$, $b=65$, $c=73$ and $d=75$. But each bag weighed more than 60 kg, so this is not the answer.

If $b+c=134$, then $b-a=2$ and hence $a=61$, $b=63$, $c=71$ and $d=77$.

Thus the weight of the lightest was 61 kg.

6. Here is the plan of a building which has a courtyard with two entrance gates. Passers-by can look through the gates but may not enter. Dimensions of the building are give in metres, and all corners are right angles. What is the area, in square metres, of that part of the courtyard which cannot be seen by passers-by?



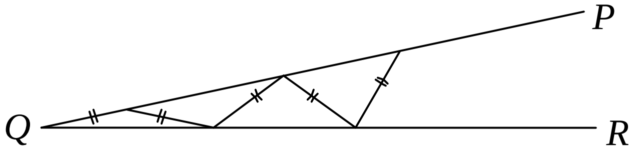
The shaded region cannot be seen by passers-by.

Since the region A is an isosceles right triangle and the length of the hypotenuse is $40-10-10=20$, the area of A is $\frac{1}{2}\times\left(\frac{20}{\sqrt{2}}\right)^2=100$.

The region B can be divided as an isosceles right triangle and a rectangle, so the area of B is $\frac{1}{2}\times\left(\frac{40}{\sqrt{2}}\right)^2+(60-40)\times 40=400+800=1200$.

Hence the area of that part of the courtyard which cannot be seen by passers-by is $1200+100=1300\text{ m}^2$.

7. In the diagram $\angle PQR=6^\circ$, and a sequence of isosceles triangles is drawn as shown. What is the largest number of such triangles that can be drawn?



Since $\angle PQR=6^\circ$, the base angle of the first isosceles triangle is 6°. Then the base angle of the second isosceles triangle is $6^\circ+6^\circ=12^\circ$, the base angle of the third isosceles triangle is $12^\circ+6^\circ=18^\circ$, an so on. Hence the base angle of the n^{th} isosceles triangle is $6n^\circ$. Note that the base angle of any isosceles triangle must less than 90°. So $6n^\circ<90^\circ$, i.e. $n<15$. Hence the largest number is 14.

8. Let $n=9+99+999+\ldots+99\ldots9$, where the last number to be added consists of 99 digits of 9. How many times will the digit 1 appear in n?

$$n=9+99+999+\cdots+\underbrace{99\cdots99}_{99\text{ digits}}$$

$$=(10-1)+(100-1)+(1000-1)+\cdots+(\underbrace{100\cdots00}_{100\text{ digits}}-1)$$

$$=\underbrace{111\cdots110}_{100\text{ digits}}-99=\underbrace{111\cdots11}_{97\text{ digits}}011$$

Hence the digit 1 appears 97+2=99 times.

9. The following multiplication example, including the answer, uses each number from 0 to 9 once and once only. Four of the numbers are filled in for you. Can you fill in the rest?

$$\begin{array}{r} \square\ 0\ 2 \\ \times\quad 3\ \square \\ \hline \square\ 5\ \square\ \square\ \square \end{array}$$

Let the multiplication be

$$\begin{array}{r} a\ 0\ 2 \\ \times\quad 3\ b \\ \hline c\ 5\ d\ e\ f \end{array}$$

Since the leading digit of the multiplier is 3 and the multiplicand is a 3-digit number, c=1, 2 or 3. Because 2 and 3 are used, c=1 and hence a<6. Thus a=4 since 1, 2, 3 and 5 are used. There are remaining 6, 7, 8 and 9.

Since 2×6=12, 2×7=14, 2×8=16 and 2×9=18, b=8 or 9.

(i) b=8:

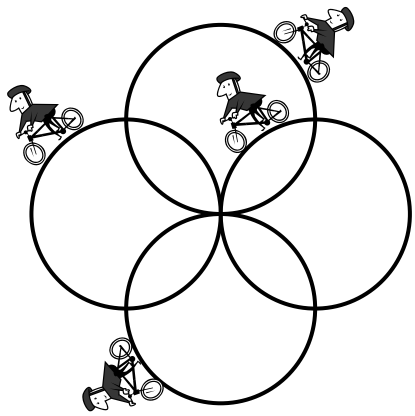
$$\begin{array}{r} 4\ 0\ 2 \\ \times\quad 3\ 8 \\ \hline 1\ 5\ 2\ 7\ 6 \end{array}$$

There are two 2's in the multiplication, so this is wrong and hence b=9.

(ii) b=9:

$$\begin{array}{r} 4\ 0\ 2 \\ \times \quad 3\ 9 \\ \hline 1\ 5\ 6\ 7\ 8 \end{array}$$

10. The four circles represent cinder paths. The four cyclists started at noon. Each person rode round a different circle, one at the rate of six miles an hour, another at the rate of nine miles an hour, another at the rate of twelve miles an hour, and the fourth at the rate of fifteen miles an hour. They agreed to ride until all met at the center, from which they started, for the fourth time. The distance round each circle was exactly one-third of a mile. When did they finish their ride?



Since one cyclist at the rate of 6 miles an hour, another at the rate of 9 miles an hour, another at the rate of 12 miles an hour, and the fourth at the rate of 15 miles an hour and each circle was exactly one-third of a mile, they could ride once round in $\frac{1}{18}$, $\frac{1}{27}$, $\frac{1}{36}$ and $\frac{1}{45}$ of an hour. They started at the center and meet at the center, so the number of round they ride must be an integer. Hence they would meet for the first time in $\frac{1}{9}$ of an hour pass noon. So they finished their ride in

$\frac{1}{9}\times 4=\frac{4}{9}$ of an hour pass noon, ie. At 12:26:40.

(Since $\frac{4}{9}$ of an hour is equal to $\frac{4}{9}\times 60=\frac{80}{3}=26\frac{2}{3}$ minutes=26 minutes and 40 seconds, the answer is at 12:26:40 as they started at 12:00:00.)