INTERNATIONL MATHENATICS AND SCIENCE OLYMPIAD FOR PRIMARY SCHOOLS (IMSO) 2005

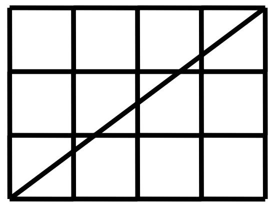
Mathematics Contest in Taiwan, Essay Problems

Name:________ School:_______ Grade:______ ID number:_________

Answer the following 10 questions, and show your detailed solution in the space provided after each question. Each question is worth 4 points.
Time limit: 60 minutes.

1. A rectangle *PQRS* with *PQ*＝45 and *PS*＝100 is cut into 4500 squares of side 1. *T* is a point on *QR* such that *QT*＝60. Of these 4500 squares, how many are cut by lines *PT* and *TS*? (A square does not cut by line if the line only passing through its vertices.)

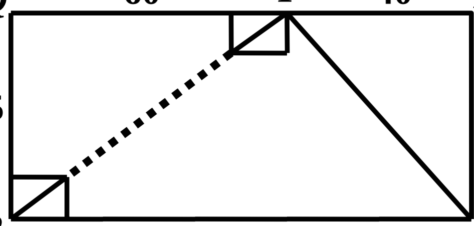
Solution:



In a 3╳4 rectangle, 6 squares are cut by its diagonal line. Since there are 15 pieces 3╳4 rectangles on the line *PT*, 15╳6=90 squares are cut by *PT*.

Similarly, in a 8╳9 rectangle, 16 squares are cut by its diagonal line. Because there are 5pieces 8╳9 rectangles on the line *TS*, 5╳16=80 squares are cut by *TS*.

Hence the total number of squares cut is then 90＋80=170.



2. In a mathematical competition consisting of 25 problems, 8 marks are given for each correct response, 0 marks for each incorrect response and each no response is awarded 3 marks. Tom scored 121 marks in this competition. What is the smallest number of incorrect responses he could have?

Solution:

Let *c* be the number of correct responses and *n* be the number of no responses. Then 8*c*+3*n*=121. Clearly, *c* can be 2, 5, 8, 11 or 14. When *c*=2, *n*=35; *c*=5, *n*=27; *c*=8, *n*=19, the total number of correct responses and no responses are greater than 25, this is impossible. When *c*=11, *n*=11 and thus the number of incorrect responses is 25-11-11=3. As *c*=14, *n*=3 and thus the number of incorrect responses is 25-14-3=8.

Hence the smallest number of incorrect responses Tom could have is 3.

3. How many numbers less than 1000 have the product of their digits equal to 63?

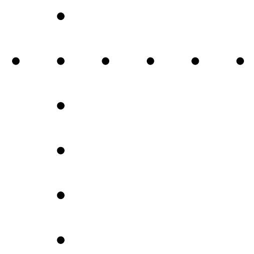
Solution:

There are no one-digit that have the product of their digits equal 63.

The digits of a two-digit number that have the product of its digits equal to 63 are 9 and 7. Hence there are two such numbers, 97 and 79.

The digits of three-digit number that have the product of its digits equal to 63 are either 1, 7, 9 or 3, 3, 7. Hence there are nine such numbers, 179, 197, 719, 791, 917, 971, 337, 373 and 733. Hence the answer is $2+9=11$.

4. How many triangles can be drawn using the points in the diagram as vertices?



Solution:

There are $6\times5\div2=15$ ways of selecting two points from the horizontal line. Each of these pairs can be matched with either of the 5 points of the orthographical line to form a triangle, so there are $5\times15=75$ ways to making a triangles using two points in the horizontal row.

Also there are $5\times4\div2=10$ ways of selecting two points from the orthographical points. Each of these pairs can be matched with either of the 5 points of the horizontal line to form a triangle, so there are $5\times10=50$ ways to making a triangle.

So the total number is $75+50=125$.

5. The integers $1, 2, 3, \cdots, 2005$ are written on the board. What is the smallest number of these integers that can be wiped off so that the product of the remaining integers ends in 8?

Solution:

In the product that remains is even, we must remove all multiples of 5, otherwise the last digit would be 0. The remaining numbers all end in 1, 2, 3, 4, 6, 7, 8 or 9.

A direct calculation shows $1\times2\times3\times4\times6\times7\times8\times9$ ends with a 6. Similarly, $11\times12\times13\times14\times16\times17\times18\times19$, $21\times22\times23\times24\times26\times27\times28\times29$, $\cdots$, $1991\times1992\times1993\times1994\times1996\times1997\times1998\times1999$ ends with a 6.

Hence the product of the remain numbers has a last digit which is the last digit of $6\times6\times\cdots\times6\times2001\times2002\times2003\times2004$, which is a 4.

Next, if we remove the number 3, the last digit of the product of remain numbers is an 8.

Thus the minimum number to be removed is $2005\div5+1=402$.

6. How many numbers less than 4000 can be formed with the digits 2, 3, 4, 5 and 6 if no digit is used more than once in a number?

Solution:

We can have five 1-digit numbers.

For the 2-digit numbers, we can have five in the first position, any of the remaining four in the second to get $5\times4=20$ such numbers.

For the 3-digit numbers, we can have five in the first position, any of the remaining four in the second and any of the remaining three in the third to get $5\times4\times3=60$ such numbers.

For the 4-digit numbers, we can have two in the first position, any of the remaining four in the next position and any of three in the third, and two in the fourth to get 2☓4☓3☓2=48 such numbers.

It gives a total of 5+20+60+48=133 numbers.

7. In a soccer tournament eight teams play each other once, with two points awarded for a win, one point for a draw and zero for a loss. How many points must a team score to that it is in the top three (i.e. has more points than at least five other teams)?

Solution:

Since there are 8 teams, must have 28 matches and thus a total of 56 points available.

Consider a team with 11 points. It is possible to have 4 teams on 11 points and 4 teams on 3 points when each of the top 4 draws with each other, each of the bottom 4 draws with each other and each of the top 4 teams wins against each of the bottom 4. So it does not guarantee a place in the top 4 if a team got 11 points.

Consider a team with 12 points. If this team was fourth then the number of points gained by the top 4 teams is greater than 48. This is impossible as the number of points shared by the bottom 4 teams is less than 8 , because these 4 teams must have at least 12 points between them for the games played between themselves. Hence 12 points is sufficient to ensure a place in top 3. Thus 12 points are required.

8. Two players take it in turns to choose from 25 numbered counters, each labeled with a different odd number from 1 to 49. When one player chooses a counter labeled X, the next player must choose the counter whose label is the greatest odd factor 0f 99—X. No matter what the first counter be taken, what is the least number of counters will remain when the game ends?

Solution:

The moves will be, when the first player chooses a number.

When the first player chooses 1, the order of the following numbers are 49, 25, 37, 31, 17, 41, 29 and 35. Since 99—35=64 has no odd factors except 1, the game ends and there are 16 counters left. We will get the same result as the first player chooses one of 49, 25, 37, 31, 17, 41, 29 and 35 because they are in the same loop.

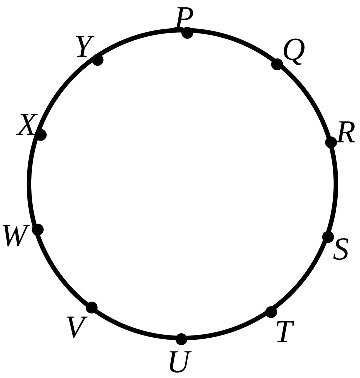
The following table is all of the situations about the first player's choices:

The first player's choice	Loop	The number of remaining counters
1, 17, 25, 29, 31, 35, 37, 41, 49	1→49→25→37→31→17→41→29→35→1	16
3	3→3	24
5, 7, 13, 19, 23, 43, 47	5→47→13→43→7→23→19→5	18
9, 27, 45	9→45→27→9	22
11	11→11	24

15, 21, 39	15→21→39→15	22
33	33→33	24

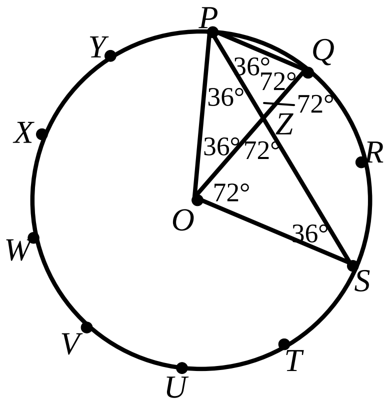
Hence there are at least 16 counters left as the game ends.

9. Ten points, P, Q, R, $\cdots$, Y, are equally spaced around a circle of radius one. What is the difference in the lengths of the lines PQ and PS?



solution:

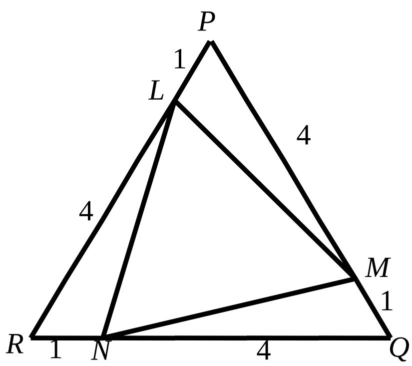
In the diagram, O is the centre of the circle and Z is the point of the intersection of the lines PS and OQ. $\angle POQ=36^\circ$ and the other angles follow as shown.



The triangles POQ and POS are isosceles, so $\angle OPQ=\angle OQP=72^\circ$ and $\angle OPS=\angle OSP=36^\circ$. Hence $\angle QPZ=36^\circ$. Thus, in $\triangle QPZ$, we have $\angle PQZ=\angle PZQ=72^\circ$. So $PS-PQ=PS-PZ=ZS=OS$ =radius of the circle=1.

10. In figure, PQR is equilateral triangle with $PR=RQ=PQ=5$. If $PL=RN=QM=1$, what is the ratio of the area of the triangle LMN to the triangle PQR?

Solution:



Consider $\triangle LRN$, $\triangle NQM$ and $\triangle LPM$. Since $\angle LRN=\angle NQM=\angle LPM=60^\circ$,

$$\frac{\triangle LRN}{\triangle PQR}=\frac{1\times 4}{5\times 5}=\frac{4}{25}.$$

Similarly, $\dfrac{\triangle NQM}{\triangle PQR}=\dfrac{4}{25}$ and $\dfrac{\triangle LPM}{\triangle PQR}=\dfrac{4}{25}$.

Hence $$\frac{\triangle LMN}{\triangle PQR}=\frac{\triangle PQR-\triangle LRN-\triangle NQM-\triangle LPM}{\triangle PQR}=\frac{25-12}{25}=\frac{13}{25}$$