International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall, 2001.**

1. An altitude of a pentagon is the perpendicular drop from a vertex to the opposite side. A median of a pentagon is the line joining a vertex to the midpoint of the opposite side. If the five altitudes and the five medians all have the same length, prove that the pentagon is regular.

2. There exists a block of 1000 consecutive positive integers containing no prime numbers, namely, $1001! + 2$, $1001! + 3$, $\ldots$, $1001! + 1001$. Does there exist a block of 1000 consecutive positive integers containing exactly five prime numbers?

3. On an east-west shipping lane are ten ships sailing individually. The first five from the west are sailing eastwards while the other five ships are sailing westwards. They sail at the same constant speed at all times. Whenever two ships meets, each turns around and sails in the opposite direction. When all ships have returned to port, how many meetings of two ships have taken place?

4. On top of a thin square cake are triangular chocolate chips which are mutually disjoint. Is it possible to cut the cake into convex polygonal pieces each containing exactly one chip?

5. The only pieces on an 8×8 chessboard are three rooks. Each moves along a row or a column without running to or jumping over another rook. The white rook starts at the bottom left corner, the black rook starts in the square directly above the white rook and the red rook start in the square directly to the right of the white rook. The white rook is to finish at the top right corner, the black rook in the square directly to the left of the white rook and the red rook in the square directly below the white rook. At all times, each rook must be either in the same row or the same column as another rook. Is it possible to get the rooks to their destinations?

Note: Each problem is worth 4 points.

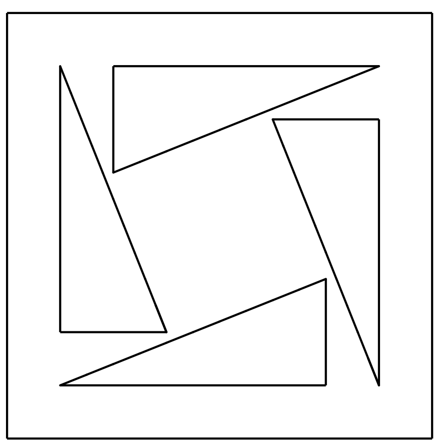
Solution to Senior O-Level Fall 2001

1. Let the pentagon be $ABCDE$. Let A', B', C', D' and E' be the respective midpoints of CD, DE, EA, AB and BC. The median AA' is at least the length of the altitude from A, and since they have the same length, AA' is also an altitude. Similarly, every median is an altitude. Now $AA' = CC'$ and $\angle AA'C = 90° = \angle CC'A$. Hence triangles ACA' and CAC' are congruent, so that $CD = 2CA' = 2AC' = EA$. Similarly, $EA = BC = DE = AB$. Hence $ABCDE$ is equilateral. The congruency of ACA' and CAC' also yields $\angle ACD = \angle EAC$, and $\angle BCA = \angle CAB$ folllows from $AB = BC$. Hence $\angle BCD = \angle EAB$, and it follows that $ABCDE$ is equiangular also. Hence it is regular.

2. Starting with the given block, we shift it back by replacing the largest number in the block with the number 1 less than the smallest number in the block. Then the number of primes in the block changes by at most 1 in each shift. By the time we shift the block to the first 1000 positive integers, the number of primes in the block is greater than 5. Thus somewhere in between, we must have hit a block of 1000 consecutive positive integers containing exactly 5 primes.

3. Let us consider what happens when two ships meet. Each continues where the other would have gone. Since we are interested in the total number of meetings rather than the numbers of meetings for individual ships, we may pretend that the ships just sail on. Since there are 5 ships from each side, the total number of meetings is $5 \times 5 = 25$.

4. This is not always possible. The diagram below shows four congruent triangles inside a square, each with a right angle, a *blunt* angle and a *sharp* angle. The line joining the centre of the square to the vertex of any sharp angle is blocked by the blunt angle of another triangle. Thus the centre cannot belong to any convex polygon containing exactly one of the triangles.



5. At all times, the rooks occupy three of the four corners of a rectangle. There are two kinds of moves. In the first, one dimension of the rectangle is changed, but the rooks occupy the same corners as before. In the second, a rook moves to the vacant corner from an adjacent one. This does not change the cyclic order of the rooks. Since the initial and final positions differ in orientation (one clockwise and one counterclockwise), the task is impossible.