International Mathematics Tournament of Towns

Autumn 2002, 20 October, Junior O-level.

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed. Points for each problem are shown in brackets [].)

1. [4] Several diagonals of a convex polygon of 2002 sides are drawn so that they do not intersect inside the polygon. As the result, the polygon splits into 2000 triangles. Can it happen that exactly half of these triangles have diagonals for all of their three sides?

2. [5] John and Mary each chose a natural number and communicated it to Bill. Bill wrote down the sum of these numbers on one piece of paper and their product on another, hid one of the pieces of paper and showed the other (on which he had written the number 2002) to John and Mary. John looked at the number and declared that he cannot say what number Mary chose. Having heard this, Mary said she doesn't know what number John chose. What was the number chosen by Mary?

3. (a) [1] An exam was conducted in a class. It is known that at least two thirds of the questions in the exam were difficult: for each such question, at least two thirds of the pupils failed to find the answer. It is also known that at least two thirds of the pupils did well on the exam: each one of these pupils answered at least two thirds of the questions. Is this possible?
   (b) [2] Will the answer to this problem be the same if two thirds is replaced by three fourths everywhere?
   (c) [2] Will the answer be the same if two thirds is replaced by seven tenths?

4. [5] 2002 cards with the numbers 1, 2, 3, ... , 2002 written on them are laid on a table face up. Two players in turn pick up one card from the table until all the cards are removed. The winner is the player for whom the last digit of the sum of the numbers on the cards chosen is the largest. Explain which of the two players can win (independently of how the other plays) and how he should play to do so.

5. [5] A certain angle with a point A inside it is given. Can three straight lines be drawn through A so that on each of the angle's sides one of the intersection points of these lines with the side be the midpoint between the intersection points of the two other lines with that same side?

International Mathematics Tournament of Towns

Autumn 2002, 20 October, Senior O-level.

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed. Points for each problem are shown in brackets [].)

1. [4] John and Mary each chose a natural number and communicated it to Bill. Bill wrote down the sum of these numbers on one piece of paper and their product on another, hid one of the pieces of paper and showed the other (on which he had written the number 2002) to John and Mary. John looked at the number and declared that he cannot say what number Mary chose. Having heard this, Mary said she doesn't know what number John chose. What was the number chosen by Mary?

2. (a) [1] An exam was conducted in a class. It is known that at least two thirds of the questions in the exam were difficult: for each such question, at least two thirds of the pupils failed to find the answer. It is also known that at least two thirds of the pupils did well on the exam: each one of these pupils answered at least two thirds of the questions. Is this possible?
(b) [1] Will the answer to this problem be the same if two thirds is replaced by three fourths everywhere?
(c) [2] Will the answer be the same if two thirds is replaced by seven tenths?

3. [5] Several straight lines, no two of which are parallel, cut up the plane into some pieces. A point A is chosen in one of these pieces. Prove that a point lying on the side opposite from A with respect to all the lines exists if and only if the piece containing A is infinite (unbounded).

4. [5] Let x, y, z be any three numbers from the open interval (0, pi/2). Prove the inequality

$$(x*\cos x + y*\cos y + z*\cos z)/(x + y + z) <= (\cos x + \cos y + \cos z)/3.$$

5. [5] In an infinite sequence of natural numbers each successive number x' is obtained from the previous number x by adding one of the nonzero digits of x to x. Prove that an even number will necessarily occur in this sequence.