International Mathematics Tournament of Towns

Autumn 2002, 27 October, Junior A-level.

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed. Points for each problem are shown in brackets [].)

1. [4] There are 2002 employees in a bank. All the employees came to celebrate the bank's jubilee and were seated around a single round table. It is known that the difference in salaries of any two employees sitting next to each other is 2 or 3 dollars. What can the maximal difference in salaries of two employees be, if it is known that all the salaries are different?

2. [5] All the species of plants existing in Russia are numbered by integers from 2 to 20000 (one after the other, without omissions or repetitions). For any pair of species, the greatest common divisor of their numbers was calculated, but the numbers themselves were lost (as the result of a computer error). Is it possible to recover the number of each of the species from that data?

3. [6] The vertices of a polygon of 50 sides divide a circle into 50 arcs, whose lengths are 1, 2, 3, ..., 50, in some order. It is known that any pair of "opposite" arcs (corresponding to opposite sides of the polygon) have lengths differing by 25. Prove that one can find two parallel sides in the polygon.

4. [6]. Point P is chosen in triangle ABC so that angle ABP equals angle ACP, while angle CBP is equal to angle CAP. Prove that P is the intersection point of the altitudes of the triangle.

5. [7] A convex polygon of N sides is divided by diagonals into triangles so that the diagonals don't intersect inside the polygon. The triangles are painted black and white so that any two triangles with common side are painted in different colors. For each N, find the maximal difference between the number of black and the number of white triangles.

6. [9] There is a large number of cards, on each of which one of the numbers 1, 2, ..., n is written. It is known that the sum of all numbers of all the cards equals k*(n!) for some integer k. Prove that the cards may be arranged in k stacks so that the sum of numbers written on the cards of each stack equals n! .

7. (a) [5] An electrical network has the shape of a three by three lattice: the network has 16 nodes (at the vertices of the 9 squares of the lattice) joined by wires (along the sides of the squares). It may have happened that some of the wires are burned out. In one measurement one can choose any pair of nodes and check if electrical current circulates between them (that is, check if there is a chain of intact wires joining the chosen nodes). Actually the network is such that current will circulate from any node to any other node. What least number of measurements is required to verify this?

   (b) [5] Same question for a network in the shape of a five by five lattice (36 nodes).

International Mathematics Tournament of Towns

Autumn 2002, 27 October, Senior A-level.

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed. Points for each problem are shown in brackets [].)

1. [4] All the species of plants existing in Russia are numbered by integers from 2 to 20000 (one after the other, without omissions or repetitions). For any pair of species, the greatest common divisor of their numbers was calculated, but the numbers themselves were lost (as the result of a computer error). Is it possible to recover the number of each of the species from that data?

2. [6] A cube is cut by a plane so that the cross-section is a pentagon. Prove that the length of one of the sides of the pentagon differs from 1 meter by at least 20 centimeters.

3. [6] A convex polygon of N sides is divided by diagonals into triangles so that the diagonals don't intersect inside the polygon. The triangles are painted black and white so that any two triangles with common side are painted in different colors. For each N, find the maximal difference between the number of black and the number of white triangles.

4. [8] There is a large number of cards, on each of which one of the numbers 1, 2, ..., n is written. It is known that the sum of all numbers of all the cards equals k*(n!) for some integer k. Prove that the cards may be arranged in k stacks so that the sum of numbers written on the cards of each stack equals n! .

5. Two circles intersect at points A and B. Through the point B a straight line is drawn, intersecting the first and second circle a second time at the points K and M, respectively. The line L_1 is tangent to the first circle at the point Q and is parallel to line AM. The line QA intersects the second circle a second time at the point R. The line L_2 is tangent to the second circle at the point R. Prove that
   (a) [4] L_2 is parallel to AK;
   (b) [4] the line L_1, L_2 and KM have a common point.

6. [8]. Consider a sequence whose first two terms equal 1 and 2, respectively, and each subsequent term is the smallest positive integer which has not yet occurred in the sequence and is not coprime with the previous term of the sequence. Prove that all positive integers occur in this sequence.

7. (a) [4] An electrical network has the shape of a three by three lattice: the network has 16 nodes (at the vertices of the 9 squares of the lattice) joined by wires (along the sides of the squares). It may have happened that some of the wires are burned out. In one measurement one can choose any pair of nodes and check if electrical current circulates between them (that is, check if there is a chain of intact wires joining the chosen nodes). Actually the network is such that current will circulate from any node to any other node. What least number of measurements is required

to verify this?
(b) [5] Same question for a network of the shape of a seven by seven lattice (64 nodes).