International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2002.**

1. Let $a < b$ be positive integers. Are their values uniquely determined if both a 49×51 rectangle and a 99×101 rectangle can be tiled by $a \times b$ rectangles?

2. Can any triangle be dissected into four convex polygons: a triangle, a quadrilateral, a pentagon, and a hexagon?

3. Let x and y be positive integers. If $x^2 + xy + y^2$ is a multiple of 10, prove that it must actually be a multiple of 100.

4. The sides AB, BC, CD and DA of a quadrilateral $ABCD$ are tangent to a circle at the points K, L, M and N respectively. Let S be the point of intersection of KM and LN. Prove that if $SKBL$ is a cyclic quadrilateral, then so is $SNDM$.

5. (a) There are 128 coins of two different weights, 64 of each. How can one find two different coins by performing no more than 7 weighings on a balance?

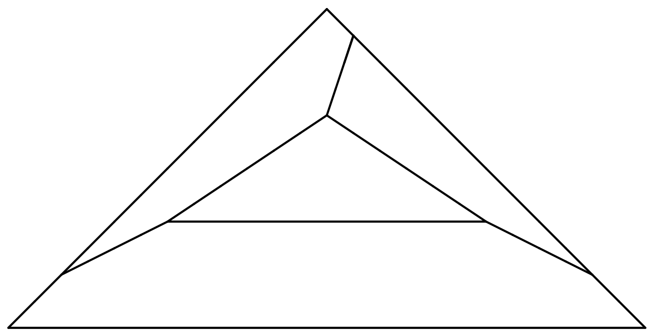
   (b) There are 8 coins of two different weights, 4 of each. How can one find two different coins by performing 2 weighings on a balance?

Note: The problems are worth 4, 5, 5, 5 and 3+3 points respectively.

Solution to Junior O-Level Spring 2002

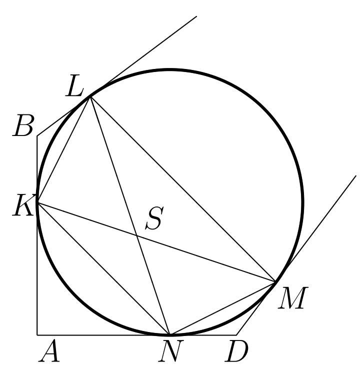
1. The only common divisors of $49 \times 51 = 3 \times 7^2 \times 17$ and $99 \times 101 = 3^2 \times 11 \times 101$ are 1 and 3. Since $a < b$, $ab > 1$. So $ab = 3$ and we must have $a = 1$ and $b = 3$.

2. One such dissection is shown in the diagram below.



3. If either x or y is odd, $x^2 + xy + y^2$ is also odd. Hence they are both even. If one is a multiple of 10 and the other is not, $x^2 + xy + y^2$ is not a multiple of 10. Suppose both x and y are not multiples of 10. Then x^2 and y^2 end in 4 or 6, while xy cannot end in 0. So we cannot have one ending in 4 and the other in 6. If x^2 and y^2 both end in 4 or both end in 6, then xy must also end in 4 or 6. It follows that the only possibility is for both x and y to be multiples of 10, so that $x^2 + xy + y^2$ will indeed be a multiple of 100.

4. Since BK and BL are tangents, $\angle BKL = \angle KML = \angle BLK$. Denote their common value by θ. Then $\angle KBL = 180° - 2\theta$. Similarly, $\angle DMN = \angle MLN = \angle DNM$. Denote their common value by ϕ. Then $\angle MDN = 180° - 2\phi$. Now $\angle KSL = \angle SLM + \angle SML = \theta + \phi$. Similarly, $\angle MSN = \theta + \phi$. Since $SKBL$ is cyclic, $\angle KBL + \angle KSL = 180°$, which implies that $\theta = \phi$. Then $\angle MDN + \angle MSN = 180°$, which implies that $SMDN$ is cyclic.



5. (a) Weigh 64 of the coins against the other 64. If they balance, discard one set. Weigh 32 of the remaining ones against the other 32, and continue. If they always balance, then after 6 weighings, we are down to 2 coins which must consist of a heavy one and a light one. Suppose balance is not achieved somewhere along the way. We may as well assume that it occurs at the first weighing. In the second weighing, weigh 32 coins from the heavier side against 32 coins from the lighter side. If they balance, discard this 64 coins. If not,

discard the 64 coins not involved in the second weighing. Continuing this way, we will be down to 2 coins after 7 weighings. They must consist of a heavy one and a light one.

(b) Weigh 4 of the coins against the other 4. If they balance, discard one set. Weigh 2 of the remaining 4 coins against the other 2. If they balance, take both coins from one side. If not, take 1 coin from each side. Suppose one side is heavier in the first weighing. Weigh 2 of these coins against the other 2. If they balance, all 4 are heavy. Take 1 of them and 1 from the lighter side in the first weighing. If they do not balance, then the heavier side consists of 2 heavy coins while the lighter side consists of 1 heavy and 1 light coin. We can accomplish the task by taking the 2 coins on the lighter side.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2002.**

1. Let x and y be positive integers. If x^2+xy+y^2 is a multiple of 10, prove that it must actually be a multiple of 100.

2. Triangles ABC and $A'B'C'$ are congruent but opposite in orientation. Prove that the mid-points of AA', BB' and CC' are collinear.

3. There are 6 pieces of cheese of different weights, numbered from 1 to 6 in ascending order of weight. It is known that three pieces may be chosen so that their total weight is equal to the total weight of the other three. How can we make such a choice by performing two weighings on balance?

4. In how many ways can we place the numbers 1 to 100 in a 2×50 table so that any two consecutive numbers are placed in squares with a common side.

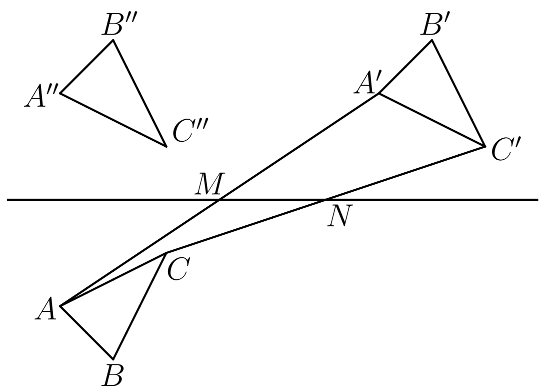
5. Does there exist a regular triangular prism that can be covered without overlap by equilateral triangles all of different sizes, if one is allowed to bend the triangles around the edges of the prism?

Note: The problems are worth 4, 5, 5, 5 and 6 points respectively.

Solution to Senior O-Level Spring 2002

1. If either x or y is odd, x^2+xy+y^2 is also odd. Hence they are both even. If one is a multiple of 10 and the other is not, x^2+xy+y^2 is not a multiple of 10. Suppose both x and y are not multiples of 10. Then x^2 and y^2 end in 4 or 6, while xy cannot end in 0. So we cannot have one ending in 4 and the other in 6. If x^2 and y^2 both end in 4 or both end in 6, then xy must also end in 4 or 6. It follows that the only possibility is for both x and y to be multiples of 10, so that x^2+xy+y^2 will indeed be a multiple of 100.

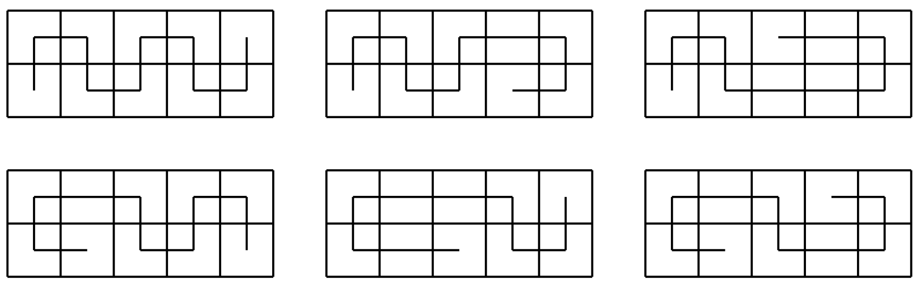
2. Let M be the midpoint of AA' and N be the midpoint of CC'. Then A and A' are equidistant from MN, as are C and C'. Let $A''B''C''$ be the reflection of ABC across MN. Then A and A'' are equidistant from MN, as are C and C''. Hence $A'A''$ and $C'C''$ are both parallel to MN. Now $A''B''C''$ is congruent to ABC and opposite in orientation. Hence it is congruent to $A'B'C'$ and in the same orientation. It follows that $A'B'C'$ and $A''B''C''$ may be obtained from each other by a translation in the direction parallel to MN. Hence B' and B'' are equidistant from MN. It follows that so are B and B', so that the midpoint of BB' indeed lies on MN.



3. The only possible groupings are (126,345), (136,245), (146,235), (156,234) and (236,145). First weigh 146 against 235. If they balance, the task is accomplished. If 146 is heavier, then 156 will be heavier than 234. Then we weigh 136 against 245. If they balance, the task is accomplished. If 136 is heavier, then 236 will be heavier than 145. Hence 126 must balance 345. If in the first weighing 146 is lighter, then 136 will be lighter than 245, 126 will be lighter than 345 and 145 will be lighter than 236. Hence 156 must balance 234.

4. We first solve the problem for a 2×5 table. Each successful placement of the numbers is replaced with a continuous path from one number to the next. Suppose first that 1 and 10 are also adjacent, so that the path could have linked up to form a cycle. The cycle could be broken up in any of 10 places. Hence there are 10 paths of this kind. Suppose now that 1 and 10 are not adjacent, so that we have an open path. We classify them according to whether the vertical segments are in one, two or three groups, where vertical segments on adjacent columns are considered to be in the same group. Note that apart from a path obtained from the cycle, each end column must contain a vertical segment.

For paths in which all the vertical segments are in one group, this means that each column must contain a vertical segment. This path, shown in the first figure below, is unique if we assume for now that the left endpoint must be on the bottom row. For paths in which the vertical segments are in two groups, we cannot have each groups containing at least two segments. On the other hand, if each contains exactly one segment, then we have a path obtainable from the cycle. Hence exactly one group contains exactly one segment. Continuing to assume that the left endpoint is on the bottom row, we have four paths as shown in the next four figures. Finally, for paths in which the vertical segments are in three groups, each end group must contain exactly one segment. This unique path is shown in the last figure. Lifting the restriction that the left endpoint be on the bottom row, we have 12 paths. Along with the 10 obtained from the cycle, we have a total of 22. Since each path may be traversed in either direction, there are 44 different placements of the numbers.



We now solve the given problem. There are 100 paths obtainable frmo the cycle. Among the others, there are 2 in which all vertical segments are in one group. For those in two groups, the larger group may consist of 2 to 48 segments. Since the larger group may be on either end, and the left endpoint may be on either row, there are $4 \times 47 = 188$ paths of this type. Finally, for those in three groups, the middle group may consist of 1 to 46 segment. For $1 \leq k \leq 46$, these segments may have $46 - k$ possible locations. Since the left endpoint may be on either row, the total number of paths of this type is $2(46 + 45 + \cdots + 1) = 2162$. Thus the total number of paths is $100 + 2 + 188 + 2162 = 2452$, and the total number of different placements of the number is $2 \times 2452 = 4904$.

5. The diagram below shows a regular triangular prism covered without overlap by three equilateral triangles of different sizes.