**International Mathematics**
**TOURNAMENT OF THE TOWNS**

**Junior A-Level Paper** **Spring 2002.**

1. Let $a,\ b$ and $c$ be the sides of a triangle. Prove that $a^3 + b^3 + 3abc > c^3$.

2. A game is played on a $23 \times 23$ board. The first player controls two white chips which start in the bottom-left and the top-right corners. The second player controls two black ones which start in the bottom-right and the top-left corners. The players move alternately. In each move, a player can move one of the chips under control to a vacant square which shares a common side with its current location. The first player wins if the two white chips are located on two squares sharing a common side. Can the second player prevent the first player from winning?

3. Let $E$ and $F$ be the respective midpoints of sides $BC$ and $CD$ of a convex quadrilateral $ABCD$. Segments $AE$, $AF$ and $EF$ cut $ABCD$ into four triangles whose areas are four consecutive positive integers. Determine the maximal area of triangle $BAD$.

4. There are $n$ lamps in a row, some of which are on. Every minute, all the lamps already on will go off. Those which were off and were adjacent to exactly one lamp that was on will go on. For which $n$ can one find an initial configuration of which lamps are on, such that at least one lamp will be on at any time?

5. An acute triangle was dissected by a straight cut into two pieces which are not necessarily triangles. Then one of the pieces was dissected by a straight cut into two pieces, and so on. After a few dissections, it turned out that all the pieces are triangles. Can all of them be obtuse?

6. In a strictly increasing infinite sequence of positive integers, every term starting from the 2002-nd term divides the sum of all preceding terms. Prove that every term starting from some term is equal to the sum of all preceding terms.

7. Some domino pieces are placed in a chain according to the standard rules. In each move, we may remove a sub-chain with equal numbers at its ends, turn the whole sub-chain around, and put it back in the same place. Prove that for every two legal chains formed from the same pieces and having the same numbers at their ends, we can transform one to the other in a finite number of moves.

**Note:** The problems are worth 4, 4, 6, 7, 7, 7 and 8 points respectively.

**Solution to Junior A-Level Spring 2002**

1. Since $b > c - a$, $a^3 + b^3 + 3abc > a^3 + (c-a)^3 + 3a(c-a)c = c^3$.

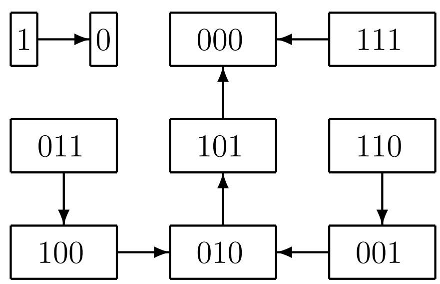
2. Initially, the four chips determine a rectangle, with chips of the same colour at opposite corners. After a move by the first player from such a position, there is no victory since the two white chips are in different rows and different columns. Moreover, the four chips will no longer determine a rectangle. However, the second player can restore this position in his move. Thus there is no victory for the first player.

3. Denote the area of the polygon $P$ by $[P]$. Then

$$[BAD] = [ABEFD] - [BEFD] = [ABE] + [AEF] + [AFD] - 3[CEF].$$

In order to maximize $[BAD]$, $CEF$ must have the smallest area among the four triangles whose area are four consecutive integers. The maximum value of $[BAD]$ is $[CEF] + 1 + [CEF] + 2 + [CEF] + 3 - 3[CEF] = 6$.

4. Denote by 0 a lamp which off and by 1 a lamp which is on. The following diagram shows that for $n = 1$ or 3, there are no initial configurations which lead to perpetual light.




For even $n$, the initial configuration 1001100110… will work since it will alternate with 0110011001…. For odd $n > 3$, just add 010 to the previous configuration. It will alternate with 100 since the third light will not go on because of the fourth. Hence this part will alternate with 100, independent of the second part. In conclusion, perpetual light is possible for all $n$ except 1 and 3.

5. A convex polygon is said to be bad if it has at least three non-obtuse angles. We claim that whenever a bad polygon is dissected into two polygons by a straight cut, at least one of the two new polygons is bad. This is because each end of the cut create at least one new non-obtuse angle, so that the two poylgons between them have at least 3+2=5 non-obtuse angles. By the Pigeonhole Principle, one of them has at least 3 of them, and is bad. This justifies the claim. Since we begin with an acute triangle, which is bad, we will always have at least one bad polygon. When all the polygons are triangles, the bad one will not be obtuse.

6. Let the sequence be $\{a_n\}$ and let $S_n$ denote the sum of all the terms up to but not including $a_n$. For $n \geq 2002$, $a_n$ is a divisor of $S_n$. Hence there exists a positive integer $d_n$ such that $a_n = \frac{S_n}{d_n}$. Then $S_{n+1} = S_n + a_n = \frac{(d_n+1)S_n}{d_n}$. If $d_{n+1} \geq d_n + 1$, then $a_{n+1} \leq \frac{S_n}{d_n} = a_n$, and this contradicts the hypothesis that $\{a_n\}$ is strictly increasing. Hence $\{d_n\}$ is non-decreasing for $n \geq 2002$. However, this sequence cannot maintain a value $k > 1$ indefinitely as otherwise $\{S_n\}$ becomes a geometric progression with common ratio $\frac{k+1}{k}$ starting from some term. However, $k$ and $k+1$ are relatively prime, and we can only divide the first term of the geometric progression by $k$ finitely many times. It follows that $d_n = 1$ eventually.

7. We use induction on the number $n$ of domino pieces in the chain. For $n = 1$ and 2, the result holds trivially. Consider the general case where the first number is $a$. Let the first piece in the initial chain be $(a, b)$ and that in the final chain be $(a, c)$. If $b = c$, we can appeal to the induction hypothesis. Assume therefore that $b \neq c$. Then the piece $(a, b)$ is now further down the chain. If it has been reversed to $(b, a)$, we simply take the sub-chain from $(a, c)$ to $(b, a)$ and reverse it. Then we appeal to the induction hypothesis. Assume therefore that $(a, b)$ has not been reversed. The proof will be complete if we can show that $(a, b)$ can be reversed. In the initial chain, let $(d, e)$ be the first piece which does not appear after $(a, b)$ in the final chain. Let the piece before $(d, e)$ in the initial chain be $(f, d)$. Then this piece appears in the final chain after $(a, b)$, possibly reversed. On the other hand, the piece $(d, e)$ appears in the final chain before $(a, b)$, also possibly reversed. We consider four possible configurations of the final chain, and verify that in each case, $(a, b)$ is reversed.
**Case 1.** $(a, c), \ldots, (d, e), \ldots, (a, b), \ldots, (f, d), \ldots$.
We reverse the sub-chain from $(d, e)$ to $(f, d)$.
**Case 2.** $(a, c), \ldots, (d, e), \ldots, (a, b), \ldots, (d, f), \ldots$.
We reverse the sub-chain from $(d, e)$ to $(g, d)$, where $(g, d)$ is the piece right before $(d, f)$.
**Case 3.** $(a, c), \ldots, (e, d), \ldots, (a, b), \ldots, (f, d), \ldots$.
We reverse the sub-chain from $(d, h)$ to $(f, d)$, where $(d, h)$ is the piece right after $(e, d)$.
**Case 4.** $(a, c), \ldots, (e, d), \ldots, (a, b), \ldots, (d, f), \ldots$.
We reverse the sub-chain from $(d, i)$ to $(j, d)$, where $(d, i)$ is the piece right after $(e, d)$ and $(j, d)$ is the piece right before $(d, f)$.

**International Mathematics**
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**Senior A-Level Paper** **Spring 2002.**

1. In triangle $ABC$, $\tan A$, $\tan B$ and $\tan C$ are integers. Find their values.

2. Does there exist a point $A$ on the graph of $y = x^3$ and a point $B$ on the graph of $y = x^3 + |x| + 1$ such that the distance between $A$ and $B$ does not exceed $\frac{1}{100}$?

3. In a strictly increasing infinite sequence of positive integers, every term starting from the 2002-th term divides the sum of all preceding terms. Prove that every term starting from some term is equal to the sum of all preceding terms.

4. The spectators are seated in a row with no empty places. Each is in a seat which does not match the spectator's ticket. An usher can order two spectators in adjacent seats to trade places unless one of them is already seated correctly. Is it true that from any initial arrangement, the usher can eventually place all the spectators in their correct seats?

5. Let $AA_1$, $BB_1$ and $CC_1$ be the altitudes of an acute triangle $ABC$. Let $O_A$, $O_B$ and $O_C$ be the respective incentres of triangles $AB_1C_1$, $BA_1C_1$ and $CA_1B_1$. Let $T_A$, $T_B$ and $T_C$ be the points of tangency of the incircle of $ABC$ with sides $BC$, $CA$ and $AB$ respectively. Prove that $T_AO_CT_BO_AT_CO_B$ is an equilateral hexagon.

6. The 52 cards in a standard deck are arranged in a $13 \times 4$ array. If every two adjacent cards, vertically or horizontally, have either the same suit or the same value, prove that all 13 cards of the same suit are in the same row.

7. Do there exist irrational numbers $a$ and $b$ such that $a > 1$, $b > 1$ and $\lfloor a^m \rfloor \neq \lfloor b^n \rfloor$ for any positive integers $m$ and $n$?

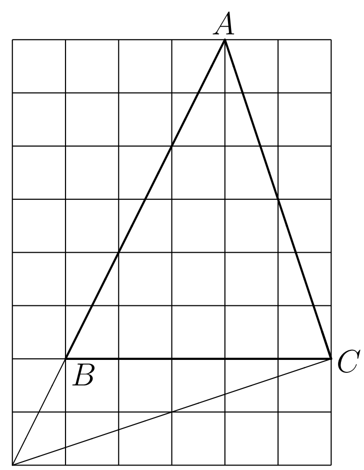
**Note:** The problems are worth 4, 4, 5, 5, 6, 7 and 8 points respectively.

## Solution to Senior A-Level Spring 2002

1. First, note that we have

$$\begin{aligned}\tan A+\tan B+\tan C &= \tan A+\tan B-\frac{\tan A+\tan B}{1-\tan A\tan B}\\ &= (\tan A+\tan B)\left(1-\frac{1}{1-\tan A\tan B}\right)\\ &= -\frac{\tan A+\tan B}{1-\tan A\tan B}\tan A\tan B\\ &= \tan A\tan B\tan C.\end{aligned}$$

Let $\tan A=a,\ \tan B=b$ and $\tan C=c$ where $a,\ b$ and $c$ are integers such that $a+b+c=abc$. $ABC$ cannot be a right triangle. Suppose $\angle A$ is obtuse. Then $a$ is negative while $b$ and $c$ are positive. If $b=c=1$, then $abc=a<a+2=a+b+c$. Any increase in the values of $b$ or $c$ will increase that of $a+b+c$ while decrease that of $abc$. It follows that $ABC$ is an acute triangle, so that $a,\ b$ and $c$ are all positive. We may assume that $a\leq b\leq c$. Then $abc=a+b+c\leq 3c$, so that $ab\leq 3$. We cannot have $a=b=1$. Hence $a=1,\ b=2$ and $c=3$. Finally, the diagram below shows a triangle $ABC$ with $\tan A=1,\ \tan B=2$ and $\tan C=3$.




2. Consider the points $A(a,a^3)$ and $B(b,b^3+b+1)$ where $a>b>0$. We wish to choose $a$ and $b$ such that $a-b<\frac{1}{100}$ while $a^3=b^3+b+1$. Let $t=a-b>0$. From $(b+t)^3=b^3+b+1$, we have $3tb^2-(1-3t^2)b-(1-t^3)=0$. If $t<\frac{1}{100}$, the constant term of this quadratic equation is negative, so that it has one positive root and one negative root. Thus $a$ and $b$ can be chosen so that $AB<\frac{1}{100}$.

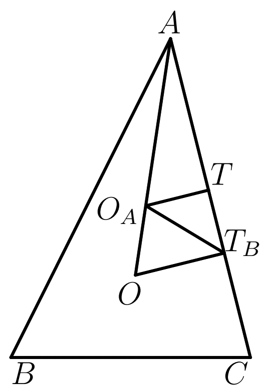
3. Let the sequence be $\{a_n\}$ and let $S_n$ denote the sum of all the terms up to but not including $a_n$. For $n\geq 2002$, $a_n$ is a divisor of $S_n$. Hence there exists a positive integer $d_n$ such that $a_n=\frac{S_n}{d_n}$. Then $S_{n+1}=S_n+a_n=\frac{(d_n+1)S_n}{d_n}$. If $d_{n+1}\geq d_n+1$, then $a_{n+1}\leq\frac{S_n}{d_n}=a_n$, and this contradicts the hypothesis that $\{a_n\}$ is strictly increasing. Hence $\{d_n\}$ is non-decreasing for $n\geq 2002$. However, this sequence cannot maintain a value $k>1$ indefinitely as otherwise $\{S_n\}$ becomes a geometric progression with common ratio $\frac{k+1}{k}$ starting from some term. However, $k$ and $k+1$ are relatively prime, and we can only divide the first term of the geometric progression by $k$ finitely many times. It follows that $d_n=1$ eventually.

4. We use induction on the number $n$ of spectators. The case $n = 2$ holds as a single switch fixes the derangement. Suppose the result holds from 1 to $n$ for some $n \geq 1$. Consider the next case with $n+1$ spectators. Let $S_k$ be the spectators with the ticket $k$. Suppose $S_{n+1}$ is in seat $m$ for some $m \leq n$. If the spectators in seats $m$ to $n+1$ constitute a derangement among themselves, we can appeal to the induction hypothesis. Otherwise, there exists a seat $\ell$ which is the first after seat $m$ to be occupied by some $S_x$ where $x \neq \ell - 1$. This means that for $m < k < \ell$, seat $k$ is occupied by $S_{k-1}$. We perform a chain of switches from seat $\ell$ back to seat $m+1$, we still have a derangement since $S_k$ is now in seat $k+2$ for $m < k < \ell$. This brings $S_x$ to seat $m+1$ and we can now switch her with $S_{n+1}$, bringing the latter one seat closer to her correct place. We can now repeat the above process until $S_{n+1}$ is in seat $n+1$, and then appeal to the induction hypothesis.

5. Since $BCB_1C_1$ is cyclic, triangles $ABC$ and $AB_1C_1$ are similar. Tthe ratio of similarity is $\cos\alpha$ where $\alpha = \angle CAB$, since $AB_1 = AB\cos\alpha$. Let $O$ be the incentre and $r$ the inradius of $ABC$, and let $T$ be the point of tangency of the incircle of $AB_1C_1$ with $AC$. Now $OT_B = r$, $O_AT = r\cos\alpha$, $AT = AT_B\cos\alpha$, $AT_B = r\cot\frac{\alpha}{2}$ and

$$TT_B = AT_B - AT = AT_B(1-\cos\alpha) = r\cot\frac{\alpha}{2}\left(2\sin^2\frac{\alpha}{2}\right) = r\sin\alpha.$$

Hence $O_AT_B = \sqrt{O_AT^2 + T_BT^2} = r$. By symmetry, the other sides of the hexagon are also equal to $r$.




6. If two adjacent cards are of the same suit, we say that there is a suit bond between them. If they are of the same rank instead, we say that there is a rank bond between them. By hypothesis, there is either a suit bond or a rank bond between two adjacent cards, and it cannot be both since each card is unique within a deck. So we have twelve columns each consisting of four horizontal bonds, and three rows each consisting of thirteen vertical bonds. We claim that in each row and each column, the bonds are of the same type. Assuming to the contrary that there are both suit bonds and rank bonds in a column. Then there is one of each kind in two adjacent rows. Of the four cards in question, let the top two be the Ace and King of Hearts. The bottom two are of the same rank. If this rank is Ace, then there is no bond between the King of Heart and the card below. Similarly, this rank cannot be King. Now not both cards at the bottom can be Hearts. Hence one of them will not have a bond with the card above. This justifies our claim. Considering the types of bonds for each of the three rows of vertical bonds, we have eight cases.

(i) All three rows are rank bonds. This yields the desired conclusion.

(ii) All three rows are suit bonds. This means that the 52 cards are in 13 groups of 4, with cards in the same group being of the same suit. This is impossible since 13 is not a multiple of 4.

(iii) Only the top and bottom rows are suit bonds. This means that we have 26 disjoint pairs of cards of the same suit. This is impossible since 13 is not a multiple of 2.

(iv) Only the top and bottom rows are rank bonds. Consider the 13 inside pairs of cards in the second and the third rows, with a suit bond between each pair. We may assume that the first pair are Spades. There must be a first pair which are not Spades, say Hearts. Consider first the subcase where the two outside cards in the first column are of the same suit, which cannot be Hearts. We may assume it is Clubs. Then the two outside cards on the column with Hearts inside must be Diamonds. When the inside pair change suits again, it must go from Hearts to either Spades or Clubs. It follows that each column of 4 cards have the same colour. However, there are 26 red cards and 26 is not a multiple of 4. Consider now the subcase where the two outside cards in the first column of different suits. Then they must be Diamonds and Clubs. Then the two outside cards on the column with Hearts inside must be Clubs and Diamonds. It follows that all the Spades and Hearts form 13 inside pairs, but there are 13 Spades and 13 is not a multiple of 2.

(v) Only the top two rows are suit bonds. We may assume that the top three cards in the first column are Spades and that the bottom card is Clubs. This remains the case until we encounter the first column of horizontal suit bonds. Then the four cards in the next column must all be red. It follows that the four cards in each column are of the same colour. However, there are 26 red cards and 16 is not a multiple of 4.

(vi) Only the bottom two rows are suit bonds. This is analogous to Case (v).

(vii) Only the top two rows are rank bonds. This means that there are 3 cards of the same rank in each column. Since there are only 4 cards of each rank, all 13 columns consist of different ranks. Hence the first row of horizontal bonds are are suit bonds, so that all columns of horizontal bonds are suit bonds. This forces all the vertical bonds in the bottom row to be rank bonds too, contrary to our assumption.

(viii) Only the bottom two rows are rank bonds. This is analogous to Case (vii).

7. Define the sequence $\{a_m\}$ by $a_0 = 2$, $a_1 = 4$ and $a_m = 4a_{m-1} - 2a_{m-2}$ for $m \geq 2$. The characteristic equation for this recurrence relation is $x^2 - 4x + 2 = 0$ and the characteristic roots are $2 \pm \sqrt{2}$. Hence $a_m = c_1(2+\sqrt{2})^m + c_2(2-\sqrt{2})^m$. From $2 = a_0 = c_1 + c_2$ and $4 = a_1 = c_1(2+\sqrt{2}) + c_2(2-\sqrt{2})$, we have $c_1 - c_2 = 0$ so that $c_1 = c_2 = 1$. Since $0 < 2-\sqrt{2} < 1$, $0 < (2-\sqrt{2})^m < 1$ for all $m \geq 1$. Hence $a_m = \lfloor a^m \rfloor$ for $m \geq 1$ where $a = 2+\sqrt{2}$. Define now the sequence $\{b_n\}$ by $b_0 = 2$, $b_1 = 6$ and $b_n = 6b_{n-1} - b_{n-2}$ for $n \geq 2$. Using the same procedure as before, we have $b_n = (3+2\sqrt{2})^n + (3-2\sqrt{2})^n$ and $b_n = \lfloor b^n \rfloor$ for $n \geq 1$, where $b = 3+2\sqrt{2}$. Now $a_m \equiv 0 \pmod 4$ for all $m \geq 1$ while $b_n \equiv 2 \pmod 4$ for all $n \geq 1$. Hence $\lfloor a^m \rfloor \neq \lfloor b^n \rfloor$ for any positive integers $m$ and $n$.