International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2008.**

1. On a 100×100 chessboard, 100 Queens are placed so that no two attack each other. Prove that if the board is divided into four 50×50 subboards, then there is at least one Queen in each subboard.
2. Each of four stones weighs an integral number of grams. Available for use is a balance which shows the difference of the weights between the objects in the left pan and those in the right pan. Is it possible to determine the weight of each stone by using this balance four times, if it may make a mistake of 1 gram either way in at most one weighing?
3. Serge has drawn triangle *ABC* and one of its medians *AD*. When informed of the ratio $\frac{AD}{AC}$, Elias is able to prove that $\angle CAB$ is obtuse and $\angle BAD$ is acute. Determine the ratio $\frac{AD}{AC}$ and justify your result.
4. Baron Münchausen asserts that he has a map of Oz showing five towns and ten roads, each road connecting exactly two cities. A road may intersect at most one other road once. The four roads connected to each town are alternately red and yellow. Can this assertion be true?
5. Let $a_1,\ a_2,\ \ldots,\ a_n$ be positive numbers such that $a_1 + a_2 + \cdots + a_n \le \frac{1}{2}$. Prove that $(1+a_1)(1+a_2)\cdots(1+a_n) < 2$.
6. *ABC* is a non-isosceles triangle. *E* and *F* are points outside triangle *ABC* such that $\angle ECA = \angle EAC = \angle FAB = \angle FBA = \theta$. The line through *A* perpendicular to *EF* intersects the perpendicular bisector of *BC* at *D*. Determine $\angle BDC$.
7. In the infinite sequence $\{a_n\}$, $a_0 = 0$. For $n \ge 1$, if the greatest odd divisor of *n* is congruent modulo 4 to 1, then $a_n = a_{n-1} + 1$, but if the greatest odd divisor of *n* is congruent modulo 4 to 3, then $a_n = a_{n-1} - 1$. The initial terms are 0, 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 3, 2 and 1.
   (a) Prove that the number 1 appears infinitely many times in this sequence.
   (b) Prove that every positive integer appears infinitely many times in this sequence.

Note: The problems are worth 4, 6, 6, 6, 8, 9 and 5+5 points respectively.