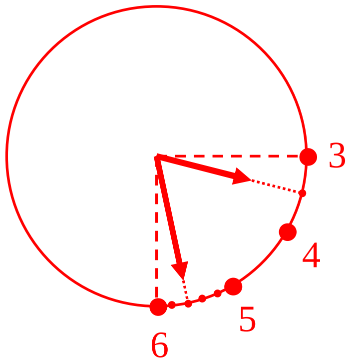
International Young Mathematicians' Convention (IYMC)
2012 Individual Contest –Junior level

1. A watch with an hour hand and a minute hand shows the time as 3:28. How many minutes later will the angle between the two hands be equal to 120°?

【Solution 】



In every minute, the minute hand moves 6°, while the hour-hand moves 0.5°. Hence the minute hand moves 5.5 °per minute more than the hour hand.
At 3:28, the angle of the minor arc between the two hands is $(5.5°\times 28-90°)=64°$.

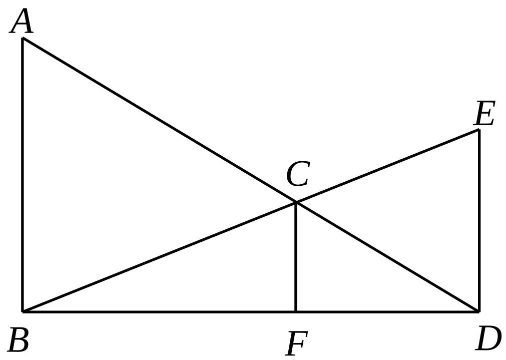
So after $(120°-64°)\div\frac{11}{2}=\frac{112}{11}=10\frac{2}{11}$ minutes, the angle between them will be 120°.

Answer: $10\frac{2}{11}$ minutes

2. In the figure below, lines *AB*, *DE* and *CF* are perpendicular to line *BD*.

If *AB* = 30 and *DE* = 20, find the length of *CF*.

【Solution】



Observe that *AB*//*DE*, so $\angle ABC=\angle DEC$ and $\angle BAC=\angle EDC$. Hence $\triangle ABC\sim\triangle DEC$.

Thus $\frac{BC}{EC}=\frac{AB}{DE}=\frac{30}{20}=\frac{3}{2}$. So $\frac{BC}{BE}=\frac{3}{3+2}=\frac{3}{5}$.

Since $\triangle BCF$ and $\triangle BED$ are similar,

$\frac{CF}{ED}=\frac{BC}{BE}=\frac{3}{5}$ and hence

$CF=\frac{3}{5}ED=\frac{3}{5}\times 20=12$.

Answer: 12

3. When 40! is expanded in base 10, what is the tenth digit from the right? (We define $n!$ as the product of the integers from 1 to n.)

【Solution】

We have $40!=2^{38}\times 3^{18}\times 5^{9}\times 7^{5}\times 11^{3}\times 13^{3}\times 17^{2}\times 19^{2}\times 23\times 29\times 31\times 37$.
There are 9 fives matched with 9 twos to give 9 zeros on the last 9 digits of the product, which is equal to 10^9 times of
$2^{29}\times 3^{18}\times 7^{5}\times 11^{3}\times 13^{3}\times 17^{2}\times 19^{2}\times 23\times 29\times 31\times 37$.
Its units digit is equal the unit digit of $2\times 9\times 7\times 1\times 7\times 9\times 1\times 3\times 9\times 1\times 7$, which is 2.
So the tenth digit from the right is 2.

Answer: 2

4. On a street, the houses are all on one side. They are numbered consecutively from 1 to 99. There is a value of *x* such that 3 times the sum of the numbers of the houses preceding the house numbered *x* is equal to the sum of the numbers of the houses following it. Find the value of *x*.

【Solution 1】

Let S_n denote the sum of the integers from 1 to *n*. From the given conditions, we have:

$$S_{x-1}=\frac{(x-1)(1+x-1)}{2}=\frac{x^2-x}{2}$$

$$S_{99}=\frac{99\times100}{2}=4950$$

$$S_x=\frac{x(1+x)}{2}=\frac{x^2+x}{2}$$

Substituting into $3S_{x-1}=S_{99}-S_x$, we have

$$3\times\frac{x^2-x}{2}=4950-(\frac{x^2+x}{2})$$

$$2x^2-x-4950=0,$$

so that *x*=50.

【Solution 2】

The sum from 1 to 99 is $\frac{99\times100}{2}=4950$.

Let the number of the house be *k*, where $1\le k\le 99$. Then $4950-k$ should be a multiple of 4. Because if the sum of the numbers of the houses preceding the house numbered *k* is *S*, then the sum of the numbers of the houses after it is 3*S*.

Hence *k*=2, 6, 10, …., 94, 98.

Now *S*= $1+2+3+\cdots+(k-1)=\frac{k(k-1)}{2}$.

$$1213=\frac{4852}{4}\le\frac{4950-k}{4}=1+2+3+\cdots+(k-1)=\frac{k(k-1)}{2}\le\frac{4948}{4}=1237$$

If *k*=51, $\frac{51\times50}{2}=1275>1237$. So *k*=51 does not satisfy the inequality.

If *k*=50, $1213<\frac{50\times49}{2}=1225<1237$. So *k*=50 satisfies the inequality.

If *k*=49, $\frac{49\times48}{2}=1176<1213$. So *k*=49 does not satisfy the inequality.

Hence *k*=50.

Answer: 50

5. The total weight of several stones is 500 kilograms. The lightest ten stones weigh 200 kilograms. The heaviest six stones weigh 145 kilograms. The weights of stones are pairwise different and are not necessary integers. Find the number of stones.

【Solution】

Let there were *n* stones.

The heaviest stone of the lightest ten stones is more then 200/10=20 kilograms.

The lightest stone of the heaviest six stones is less then $\frac{145}{6}=24\frac{1}{6}$ kilograms.

Total weight of the rest $n-16$ stones then equals 155 kilograms, so

$20<\frac{155}{n-16}<24\frac{1}{6}$ or $24(n-16)<186<29(n-16)$. Hence n=23.

An example with n=23: weights of stones are equal to

19, $19\frac{1}{5}$, $19\frac{2}{5}$, $19\frac{3}{5}$, $19\frac{4}{5}$, $20\frac{1}{5}$, $20\frac{2}{5}$, $20\frac{3}{5}$, $20\frac{4}{5}$, 21, $21\frac{5}{7}$, $21\frac{6}{7}$, 22, $22\frac{1}{7}$, $22\frac{2}{7}$, $22\frac{3}{7}$, $22\frac{4}{7}$, $23\frac{4}{6}$, $23\frac{5}{6}$, 24, $24\frac{2}{6}$, $24\frac{3}{6}$ and $24\frac{4}{6}$ kilograms respectively.

Answer: 23 stones

6. Find the number of 10-digit numbers such that each of them is formed by using each of the digits 0 to 9 once, with no digit smaller than both neighbours.

【Solution】

Observe that the digit 0 can't be the first digit and it must be the last digit, otherwise it will be between two other digits which will be larger than it. So there is only 1 possible position for 0.

There are now nine consecutive places in which to put the digits from 1 to 9. We must put the 1 at one end of the remaining places, otherwise it will be immediately between two larger digits. Hence there are two ways of doing this.

There are now eight consecutive places in which to put the digits from 2 to 9.

Continuing the argument, we have two ways of placing the 2, 3, 4, 5, 6, 7 and 8.

There is one empty space and 9 must be placed in it. Therefore, the number of 10-digit numbers that can be formed is $1\times2\times2\times2\times2\times2\times2\times2\times2\times1=2^8=256$.

Answer: 256

7. A positive integer is said to be *curious* if it is the smallest of all positive integers with the same sum of digits. If all the *curious* numbers are arranged in ascending order, what is the 100^{th} number?

【Solution】

Let n be a *curious* number, then all of its digits but first are equal to 9. If not, increasing any not first digit that less than 9 by 1 and decreasing the first digit by 1, we obtain a smaller number with the same sum of digits.

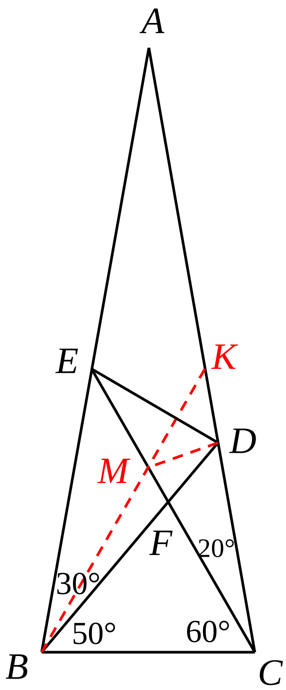
Obviously the converse is true, any number of the type $\overline{n99\cdots9}$ is *curious*.

The n-th *curious* numbers must have n as its digits sum. First nine *curious* numbers are 1, 2, ..., 9, next nine *curious* numbers are 19, 29, ..., 99, and so on.

Since $100 = 9\times11+1$, so 100^{th} *curious* number is equal to a 12-digit number 199999999999.

Answer: 199999999999

8. In triangle ABC, $AB = AC$. D is a point on AC and E is a point on AB. BD and CE intersect at point F. If $\angle DBC = 50°$, $\angle ABD = 30°$, $\angle DCE = 20°$ and $\angle ECB = 60°$, what is the measure of $\angle CED$, in degree?



【Solution】

We know that $\angle A = 20°$.

Draw line segment BK and let $\angle KBD = 10°$, which is intersecting BC at the point K and intersecting CE at point M. Then BCM is an equilateral triangle.

Connecting line segment DM. Since $\angle BDC = 50° = \angle DBC$.

Hence $BC = MC = DC$, we get DMC is an isosceles triangle and $\angle MDB = 80° - 50° = 30°$, $\angle CMD = 80°$.

We have $\angle KMD = 180° - 80° - 60° = 40°$, and also $\angle BKC = 40°$ that means $KD = MD$.

Connecting line segment EK. Since ABC is an isosceles triangle, we can get that triangles KBC and ECB are congruent, then $BK = EC$ and $KC = EB$, hence $AE = AK$.

We get that $\angle AKE = \angle AEK = 80°$.

So $\angle MEK = \angle MKE = 60°$, hence $EK = EM$.

We can get triangles EKD and EMD are congruent.

Hence $\angle MED = \angle DEK = \dfrac{60°}{2} = 30°$

Answer: 30°