INTERNATIONL MATHEMATICS AND SCIENCE OLYMPIAD FOR PRIMARY SCHOOLS (IMSO) 2006

Mathematics Contest in Taiwan, Exploration Problems

Answer

Answer the following 5 questions, and show your detailed solution in the answer sheet. Write down the question number in each paper. Each question is worth 8 points. Time limit: 60 minutes.

1. The solution to each clue of this crossnumber is a two-digit number. None of these numbers begins with zero. Complete the crossnumber, stating the order in which you solved the clues and explaining why there is only one solution.

Clues Across

1. A square number
3. A multiple of 11

Clues Down

1. A multiple of 7
2. A cube number

1.	2.
3.	

Observe that the two-digit cube numbers less than 100 are 27 and 64 and the two-digit square numbers less than 100 are 16, 25, 36, 49, 64 and 81. Since the unit digit of the square number is the leading digit of the cube number, the cube number is 64 and hence the multiple of 11 is 44. So the unit digit of the multiple of 7 is 4. And the leading digit of the square number is also a leading digit of the multiple of 7, so the multiple of 7 is 14.

1	6
4	4

2. Notice that $2^2 + 2^2 = 2^3$, so two squares can sum to give a cube; however, the two squares here are equal (to 4).
   (a) Find two unequal squares whose sum is a cube.
   (b) Show that there are infinitely many pairs of unequal squares whose sum is equal to a cube.

(a) $5^3=125=121+4=11^2+2^2$

(b) Since $(2\boldsymbol{k}^3)^2+(11\boldsymbol{k}^3)^2=(2^2+11^2)\boldsymbol{k}^6=5^3\boldsymbol{k}^6=(5\boldsymbol{k}^2)^3$, where $\boldsymbol{k}$ is a nonzero integer, there are infinitely many pairs of unequal squares, $2\boldsymbol{k}^3$ and $11\boldsymbol{k}^3$, whose sum is equal to a cube.

3. Is it possible to find a number 11…11 that is divisible by 19?

$11 \equiv 11 \pmod{19}$

$111 = 95 + 16 \equiv 16 \pmod{19}$

$1111 = 1110 + 1 \equiv 160 + 1 \equiv 161 \equiv 9 \pmod{19}$

$11111 = 1111 \times 10 + 1 \equiv 9 \times 10 + 1 \equiv 91 \equiv 15 \pmod{19}$

$111111 = 11111 \times 10 + 1 \equiv 15 \times 10 + 1 \equiv 151 \equiv 18 \pmod{19}$

$1111111 = 111111 \times 10 + 1 \equiv 18 \times 10 + 1 \equiv 181 \equiv 10 \pmod{19}$

$11111111 = 1111111 \times 10 + 1 \equiv 10 \times 10 + 1 \equiv 101 \equiv 6 \pmod{19}$

$111111111 = 11111111 \times 10 + 1 \equiv 6 \times 10 + 1 \equiv 61 \equiv 4 \pmod{19}$

$1111111111 = 111111111 \times 10 + 1 \equiv 4 \times 10 + 1 \equiv 41 \equiv 3 \pmod{19}$

$11111111111 = 1111111111 \times 10 + 1 \equiv 3 \times 10 + 1 \equiv 31 \equiv 12 \pmod{19}$

$111111111111 = 11111111111 \times 10 + 1 \equiv 12 \times 10 + 1 \equiv 121 \equiv 7 \pmod{19}$

$1111111111111 = 111111111111 \times 10 + 1 \equiv 7 \times 10 + 1 \equiv 71 \equiv 14 \pmod{19}$

$11111111111111 = 1111111111111 \times 10 + 1 \equiv 14 \times 10 + 1 \equiv 141 \equiv 8 \pmod{19}$

$111111111111111 = 11111111111111 \times 10 + 1 \equiv 8 \times 10 + 1 \equiv 81 \equiv 5 \pmod{19}$

$1111111111111111 = 111111111111111 \times 10 + 1 \equiv 5 \times 10 + 1 \equiv 51 \equiv 13 \pmod{19}$

$11111111111111111 = 1111111111111111 \times 10 + 1 \equiv 13 \times 10 + 1 \equiv 131 \equiv 17 \pmod{19}$

$111111111111111111 = 11111111111111111 \times 10 + 1 \equiv 17 \times 10 + 1 \equiv 171 \equiv 0 \pmod{19}$

So $\underbrace{111111111111111111}_{18\ \text{1s}}$ is divided by 19.

4. The menu in the school cafeteria never changes. It consists of 10 different dishes. Peter decides to make his school lunch different everyday (at least 1 dish). For each lunch, he may eat any number of dished, but no two are identical.

(a) What is the maximum numbers of days Peter can do so?

(b) What is the total number of dishes Peter has consumed during this period?

(a) Each dish has two possibilities: each dish may be chosen by Peter or not be chosen by Peter in a day. Since Peter eats at least 1 dish everyday, there are $2^{10} - 1 = 1023$ different ways and hence the maximum numbers of days is 1023.

(b) Mark the 10 dishes as A, B, C, D, E, F, G, H, I and J. Thus we can get the below table, where each row means the way that Peter eats, 1 means Peter eats the dish and 0 means Peter doesn't eat the dish at that day. So there are 1024 rows.

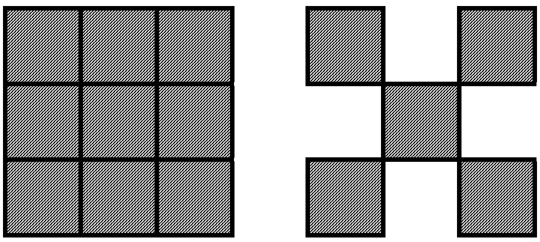
A	B	C	D	E	F	G	H	I	J
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1	0	0

⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	1	1	1	1	1	1	0	1	1
1	1	1	1	1	1	1	1	0	1
1	1	1	1	1	1	1	1	1	0
1	1	1	1	1	1	1	1	1	1

Each column has 1024÷2=512 1s, so there are 512×10=5120 1s. This means that Peter eats 5120 dishes during this period.

5. A sequence of shapes is made as follows.
   (1) Shape S_1 is a shaded square of side 1 unit.
   (2) Shape S_2 is made by dividing S_1 into 9 equal squares and removing four of these, so that only the central and corner squares remain.
   (3) Shape S_3 is made by applying the process in (2) to each of the squares of S_2.
   (4) Shape S_4 is made by applying the process in (2) to each of the squares of S_3.

   And so on.



S_1 S_2

   (a) Find the area and perimeter of shape S_3, giving your answers as fractions.
   (b) Find the least value of k for which the shape S_k has the area less than $\frac{1}{30}$ and also has perimeter greater than 30.

(a) The area of S_1=1, the area of $S_2=\frac{5}{9}$ and the area of $S_3=1\times\frac{5}{9}\times\frac{5}{9}=\frac{25}{81}$.

The perimeter of $S_1=1\times4=4$, the perimeter of $S_2=4\times\frac{5}{3}=\frac{20}{3}$ and the perimeter of $S_3=4\times\frac{5}{3}\times\frac{5}{3}=\frac{100}{9}$.

(b) We want to find k so that the area of $S_k=\left(\frac{5}{9}\right)^{k-1}\le\frac{1}{30}$ and the perimeter of $S_k=\left(\frac{5}{3}\right)^{k-1}\times4\ge30$. From (a), we know $k\ge3$.

When k=4, we know $\frac{5\times5\times5}{9\times9\times9}=\frac{125}{729}>\frac{125}{3750}=\frac{1}{30}$ and

$\frac{5\times5\times5}{3\times3\times3}\times4=\frac{500}{27}<\frac{810}{27}=30$.

When k=5, we know $\frac{5\times5\times5\times5}{9\times9\times9\times9}=\frac{625}{6561}>\frac{625}{19750}=\frac{1}{30}$ and

$\frac{5\times5\times5\times5}{3\times3\times3\times3}\times4=\frac{2500}{81}>\frac{2430}{81}=30$. So the perimeter of S_k=$\left(\frac{5}{3}\right)^{k-1}\times4\geq30$ as $k\geq5$.

When k=6, we know $\frac{5\times5\times5\times5\times5}{9\times9\times9\times9\times9}=\frac{3125}{59049}>\frac{3125}{93750}=\frac{1}{30}$.

When k=7, we know $\frac{5\times5\times5\times5\times5\times5}{9\times9\times9\times9\times9\times9}=\frac{15625}{531441}<\frac{15625}{468750}=\frac{1}{30}$

So the area of S_k=$\left(\frac{5}{9}\right)^{k-1}\leq\frac{1}{30}$ when $k\geq7$.

Hence the least value of k for which the shape S_k has the area less than $\frac{1}{30}$ and also has perimeter greater than 30 is 7.