Senior Division

1. $2(5.61 - 4.5) = 2(1.11) = 2.22$,

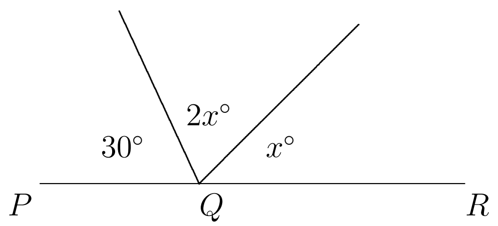
hence (D).

2.

$$\begin{aligned} 2^n + 2^n &= 2^m \\ 2(2^n) &= 2^m \\ 2^{n+1} &= 2^m \\ n+1 &= m, \end{aligned}$$

hence (B).

3. Since PQR is a straight line, $30 + 2x + x = 180$, $3x = 150$



and $x = 50$,

hence (C).

4. (Also J10 & I5)
We can see that $\frac{7}{15}$, $\frac{3}{7}$ and $\frac{4}{9}$ are each less than $\frac{1}{2}$, (E) is $\frac{1}{2}$ and $\frac{6}{11} > \frac{1}{2}$, so $\frac{6}{11}$ is the largest,

hence (C).

5. (Also I7)
Since $7 \times 89 = 623$, the length of the call was 7 minutes, and 7 minutes from 10:57 am is 11:04 am and so the call finished then,

hence (C).

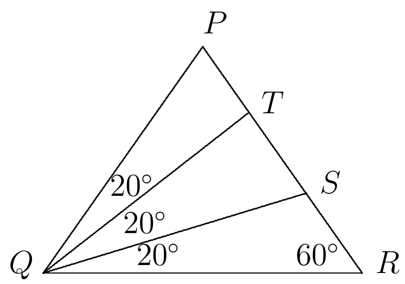
6. Since $(2, k)$ lies on the 2 lines, we have

$$4 + k = q \quad \text{and} \quad 4 = 2 - p.$$

So, $4 + (2 - p) = q$ and $p + q = 6$,

hence (E).

7. (Also J17)
As QT and QS trisect $\angle PQR$ we get the angles as shown in the diagram.



Then $\angle QST = 80°$ (exterior angle) and so, from the angle sum of $\triangle QTS$, $\angle QTS = 180° - 100° = 80°$,

hence (C).

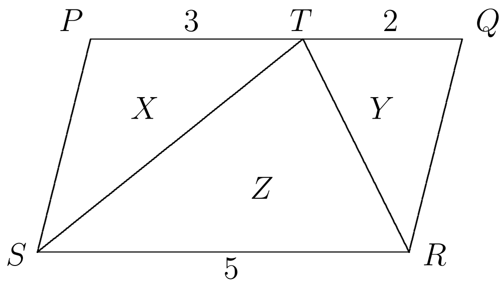
8. (Also J20)
The possibilities are

(A) 13 & 14. $14 = 2 \times 7$ which has 4 factors.
(B) 19 & 20. $20 = 2^2 \times 5$ which has 6 factors.
(C) 37 & 38. $38 = 19 \times 2$ which 4 factors.
(D) 43 & 44. $44 = 2^2 \times 11$ which has 6 factors.
(E) 53 & 54. $54 = 2 \times 3^3$ which has 8 factors.

Andy's age has 8 factors,

hence (E).

9. (Also I14)
Let the areas of the triangles PTS, TQR and RST be X, Y and Z respectively.



These triangles have the same height h, so $X = \frac{3h}{2}$, $Y = \frac{2h}{2}$ and $Z = \frac{5h}{2}$.
So

$$\frac{X+Z}{X+Y+Z} = \frac{\frac{3h}{2} + \frac{5h}{2}}{\frac{3h}{2} + \frac{2h}{2} + \frac{5h}{2}} = \frac{4}{5},$$

hence (D).

10. The numbers must be $x, y, 5, 8, 8$ where $y > x$ to have a median of 5 and only mode of 8.

$$\begin{aligned} \text{Mean} &= \frac{1}{5}(x+y+21) = 5 \\ \therefore \quad x+y+21 &= 25 \\ x+y &= 4 \end{aligned}$$

So $x = 1$ and $y = 3$ as $y > x$.
The difference between the largest and smallest is $8 - 1 = 7$,

hence (D).

11. The recommended dose is 4 drops per litre.
So, after he used the first 2 litres, the remaining 6 litres contained 12 drops.
When he filled it up again, he had 8 litres with the 12 drops, so he needed to add 20 more drops,

hence (A).

12. (Also J13 & I12)
Using the rules that each row, column and samll square contains 1, 2, 3 and 4, we fill in the squares one at a time, then

	2		
			1
	1	3	
4			

$\longrightarrow$

1	2		
			1
	1	3	
4	3		

$\longrightarrow$

1	2		
	4		1
2	1	3	
4	3		

$\longrightarrow$

1	2		3
3	4		1
2	1	3	4
4	3		

$\longrightarrow$

1	2	4	3
3	4	2	1
2	1	3	4
4	3	1	2

The sum of the numbers in the four corners is $1 + 4 + 2 + 3 = 10$,

hence (E).

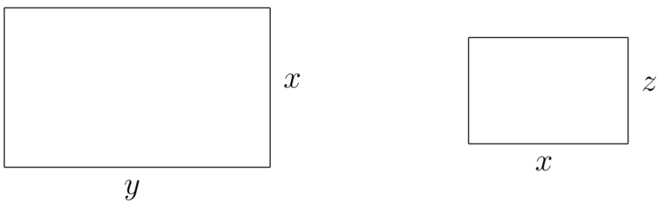
13. Ho;lly writes down 11, 13, 17, 19, 31, 33, 37, 39, 71, 73, 77, 79, 91, 93, 97, 99.

The primes are 11, 13, 17, 19, 31, 37, 71, 73, 79 and 97.

So there are 10 primes in the sample space of 16, and the probability of a prime number is $\frac{10}{16} = \frac{5}{8}$,

hence (A).

14. Let the two rectangles have sides y, x and x, z where $y > x > z$.



As the combined area is 40 cm^2, we have $5 \geq x \geq 2$, since if $x = 6$ the area of the large rectangle would be > 40 and if $x = 1$ the smaller rectangle would not exist.

Since the perimeter of the larger rectangle is twice that of the smaller, we get

$$\begin{aligned} x + y &= 2(x + z) \\ z &= \frac{y - x}{2} \end{aligned}$$

The combined areas are $40\,\text{cm}^2$, so

$$\begin{aligned} yx + xz &= 40 \\ xy + \frac{x(y - x)}{2} &= 40 \\ 3xy - x^2 &= 80 \\ x(3y - x) &= 80. \end{aligned}$$

Since x divides 80, we consider $x = 2, 4, 5$.
$x = 2$ gives $3y = 42$, not possible.

$x = 4$ gives $3y = 24$, $y = 8$, possible.
$x = 5$ gives $3y = 21$, $y = 7$, possible.

When $x = 4$, $y = 8$ and $z = 2$, and this results in two similar rectangles.

The only solution is when $x = 5$, $y = 7$ and $z = 1$,

hence (A).

15. Let the two-digit number be $10x + y$.
Reversing the digits gives $10y + x$.
Thus $10x + y + 10y + x = 11(x + y)$ where $1 \leq x + y \leq 18$.

For $11(x + y)$ to be a perfect square, $x + y = 11$, and the possible numbers are 29, 38, 47, 56, 65, 74, 83 and 92, eight numbers in all,

hence (D).

16. Number the seats from 1 to 6.

1	2	3	4	5	6	
A		B		C		6
A		B			C	6
A			B		C	6
	A		B		C	6

We can see that there are 4 different combinations of three seats possible, and for each of these we can place Ann, Bill and Carol in 6 different ways, giving $4 \times 6 = 24$ different ways,

hence (B).

17. Consider the equation $a^b = 1$.

There are three cases, case 1 when $b = 0$ and $a \neq 0$; case 2 when $a = 1$; and case 3 when $a = -1$ and b is an even integer.

Case 1: When $x + 1 = 0$, $x = -1$ and $x^2 - 3x + 1 = 5 \neq 0$, so $x = -1$ is a solution.

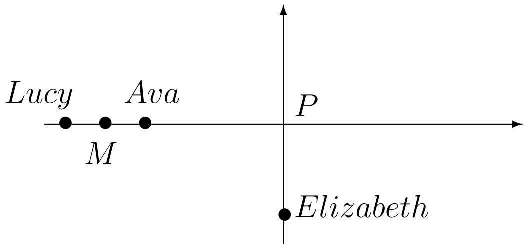
Case 2: $x^2 - 3x + 1 = 1$, $x^2 - 3x = 0$ and $x(x - 3) = 0$ so $x = 0$ or 3, gives 2 solutions.

Case 3: $x^2 - 3x + 1 = -1$, $x^2 - 3x + 2 = 0$, $(x - 2)(x - 1) = 0$, $x = 2$ and 1. But $x + 1$ must be even, and it is not for $x = 2$ but is for $x = 1$.

So the solutions are $x = -1$, 0, 1, and 3,

hence (D).

18. Since Ava and Lucy jog along the same path at 8 km/h, the mid-point M between



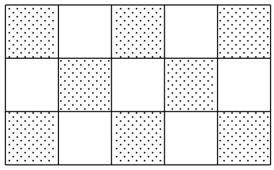
Lucy and Ava moves at 8 km/h and is 56 m from P when Ava is 50 m from P. M travels 56 m whilst Elizabeth travels

$$\frac{6}{8} \times 56 = 42\,\text{m},$$

hence (B).

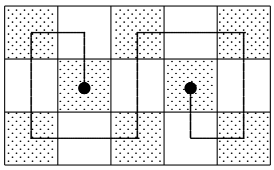
19. (Also J24)

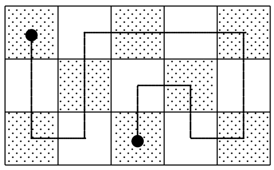
Colour the 15 squares black and white in the usual chessboard fashion with dark squares at all the corners. Then there are 8 black squares and 7 white squares.



Since the squares alternate in colour along the counter's path, the starting square and the end square have to be black. Hence it is impossible for any of the 7 white squares to be starting squares.

There are only three non-equivalent positions of black squares, the corner ones, the interior ones and the ones on the edge but not in a corner. The following two paths show that any black square could have been a starting square.

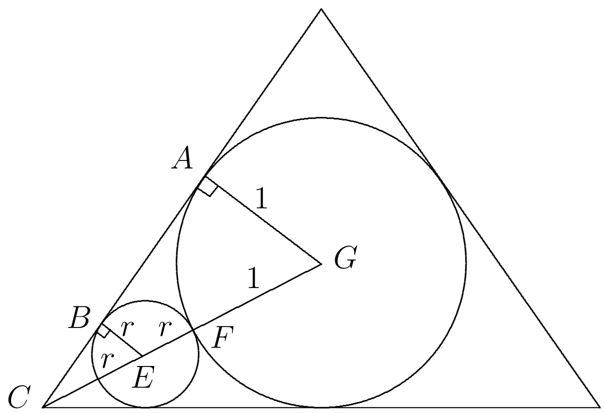




So there are 8 starting squares,

hence (D).

20. Let the radius of the small circle be r. Both $\triangle AGC$ and $\triangle BEC$ are $90°$, $60°$, $30°$ triangles with sides in the ratio $2 : \sqrt{3} : 1$.



Hence $GC = 2$ and $CE = 2r$, but

$$\begin{aligned} GC &= GF + FE + CE \\ 2 &= 1 + r + 2r \\ r &= \frac{1}{3}, \end{aligned}$$

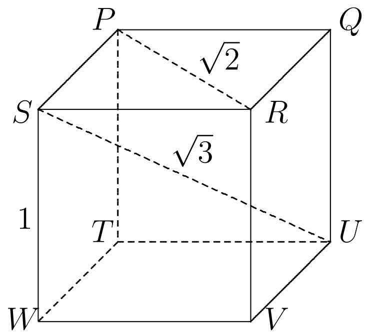
hence (A).

21. (Also I24 & J30)
There are twelve lift stops altogether. each pair of floors has a lift which connects them. Hence, as $\binom{6}{2} = 15 > 12$, there are at most five floors.
This is possible if the first lift stops on floors 1, 4 and 5, the second on 2, 4 and 5, the third in 3, 4 and 5, and the fourth on 1, 2 and 3.

So, the maximum number of floors is 5,

hence (B).

22. (Also I25)
Consider the cube $PQRSTUVW$.

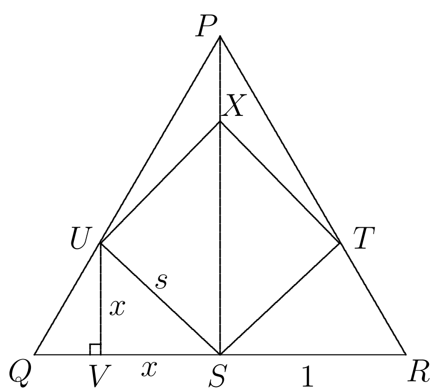


Since the bee flew so that it visited every vertex of the cube without being twice at the same point, the bee's path consists of exactly 7 straight line segments. The length of an edge is 1 unit, the length of a diagonal of a face is $\sqrt{2}$ and the length of a diagonal of the cube is $\sqrt{3}$.

The bee's path cannot have more than one diagonal of the cube as any two of them meet in the centre of the cube. So the largest possible length of such path is at most $\sqrt{3} + 6\sqrt{2}$. The example $PRUWQTSV$ shows that a path of such length does exist.

hence (D).

23. (Alternative 1)
Draw UV perpendicular to QR. Join PS. Let $UC = x$ and the side of the square be s.



Now, as PS bisects $\angle UST$, $\angle USV = \angle VUS = 45°$ and then $VS = x$, so $QV = 1 - x$.

Now, the triangles QVU and QSP are similar (equiangular) and so

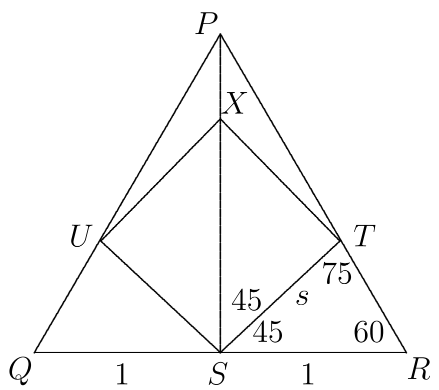
$$\begin{aligned}
\frac{QV}{QS} &= \frac{VU}{SP} \\
\frac{x}{\sqrt{3}} &= \frac{1-x}{1} \\
x &= \sqrt{3} - x\sqrt{3} \\
x &= \frac{\sqrt{3}}{1+\sqrt{3}} = \frac{3-\sqrt{3}}{2}
\end{aligned}$$

Now, from the triangle VUS, we get

$$\begin{aligned}
s^2 &= 2x^2 = 2\frac{(3-\sqrt{3})^2}{4} \\
&= \frac{12-6\sqrt{3}}{2} = 6-3\sqrt{3},
\end{aligned}$$

hence (A).

(Alternative 2)
Join PS. Clearly $\angle TRS = 60^\circ$ and $\angle TSR = 45^\circ$, hence $\angle STR = 75^\circ$.



By the sine rule, $\dfrac{s}{\sin 60^\circ} = \dfrac{1}{\sin 75^\circ}$.
Now, $\sin(A+B) = \sin A \cos B + \cos A \sin B$, so

$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \dfrac{1}{\sqrt{2}} \times \dfrac{1}{2} + \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

So, $s = \dfrac{\sin 60^\circ}{\sin 75^\circ} = \dfrac{\sqrt{3}/2}{(\sqrt{3}+1)/2\sqrt{2}} = \dfrac{\sqrt{6}}{\sqrt{3}+1}$

and the area $s^2 = \dfrac{6}{4+2\sqrt{3}} = \dfrac{3}{2+\sqrt{3}} = \dfrac{3(2-\sqrt{3})}{4-3} = 6-3\sqrt{3}$,

hence (A).

24. Given $f(x) = ax^2 + bx + c$ and $f(x)f(-x) = f(x^2)$ for all x, we get

$$(ax^2+bx+c)(ax^2-bx+c) \equiv ax^4 + bx^2 + c.$$

This gives $a^2 = a$, $2ac - b^2 = b$ and $c^2 = c$, and $a = 0$ or 1, $c = 0$ or 1.

Case 1: $a = 0$. Then $b = 0, -1$, $c = 0, 1$ and $f(x) = 0, 1, -x, 1-x$.

Case 2: $a = 1$, $c = 0$, $b = 0, -1$ and $a = c = 1$ and $b = 1, -2$. These give $f(x) = x^2, x^2 - x$; and $f(x) = x^2 + x + 1, x^2 - 2x + 1$.

So there are 8 such functions,

hence (C).

25. (Alternative 1)

$$\begin{aligned}
(\sqrt{2}+1)^1 &= \sqrt{2}+1 \\
(\sqrt{2}+1)^2 &= 2\sqrt{2}+3 \\
(\sqrt{2}+1)^3 &= 5\sqrt{2}+7 \\
(\sqrt{2}+1)^4 &= 12\sqrt{2}+17 \\
(\sqrt{2}+1)^5 &= 29\sqrt{2}+41 \\
(\sqrt{2}+1)^6 &= 70\sqrt{2}+99 \\
(\sqrt{2}+1)^7 &= 169\sqrt{2}+239 \\
(\sqrt{2}+1)^8 &= 408\sqrt{2}+577 \\
(\sqrt{2}+1)^9 &= 595\sqrt{2}+1393
\end{aligned}$$

Modulo 3. the coefficientsbehave as

1,1; 2, 0; 2, 1; 0, 2; 2, 2; 1, 0; 1, 2; 0, 1; 1, 1

This pattern clearly repeats every 8 powers ($2007 = 8 \times 250 = 7$).

Hence, if $(\sqrt{2}+1)^{2007} = a + b\sqrt{2}$ then $b \equiv 1 \pmod 3$.
So, the highest common factor of b and 81 is 1,

hence (A).

(Alternative 2)
Since $(\sqrt{2}+1)^{2007} = a + b\sqrt{2}$, we have $(\sqrt{2}-1)^{2007} = b\sqrt{2} - a$.

By multiplying these two equations we obtain

$$(\sqrt{2}+1)^{2007}(\sqrt{2}-1)^{2007} = (a+b\sqrt{2})(b\sqrt{2}-a).$$

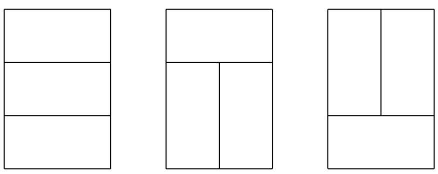
Hence $1 = 2b^2 - a^2$.

If b is divisible by 3, then a^2 is congruent -1 modulo 3, which is not possible,

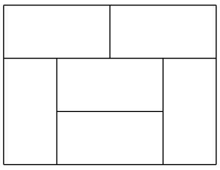
hence (A).

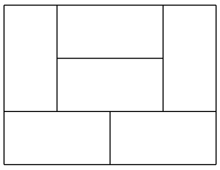
26. A 3 by 6 rectangle can be broken down into three 3 by 2 rectangles, one 3 by 4 rectangle and one 3by 2 rectangle, or one 3 by 6 rectangle.

For the 3 by 2 rectangles we have 3 tilings as shown.

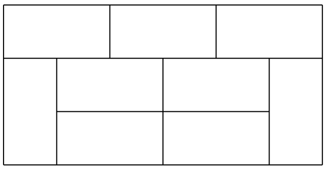


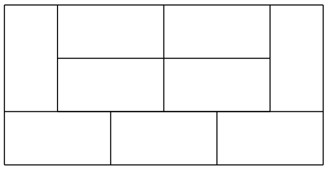
For the 3 by 4 rectangle, we can obtain 9 tilings by combining two tilings for the 3 by 2 rectangles. However, there are also 2 solutions which cannot be obtained this way. These are called irreducible tilings and they are shown below.





For the 3 by 6 rectangle, we also have 2 irreducuble tilings as shown below.





By combining the three different tilings to the 3 by 2 rectangles we get $3\times3\times3 = 27$ different tilings.

By combining a tiling for the 3 by 2 rectangle with an irreducible 3 by 4 rectangle we get $2 \times 3 \times 2 = 12$ different tilings (we can put the 3 by 2 at either end of the 3 by 4).

Adding the 2 irreducibletilings for the 3 by 6 rectangle, we get $2 + 12 + 27 = 41$ different tilings.

27. Since the distance from P_i to P_{i+1} is $\dfrac{1}{i}$ we get the distances between successive points to be

P_1P_2 occurs in 1×41 distances: P_1P_2, P_1P_3, P_1P_4 and so on.
P_2P_3 occurs in 2×40 distances: $P_1P_3, P_2P_3; P_1P_4, P_2P_4$ and so on.
P_3P_4 occurs in 3×39 distances.
$\vdots$
$P_{41}P_{42}$ ocurs in 41×1 distances.

Hence the sum of all these distances

$$\begin{aligned} &= 41 \times 1 \times 1 + 40 \times 2 \times \frac{1}{2} + 39 \times 3 \times \frac{1}{3} + \cdots + 41 \times 1 \times \frac{1}{41} \\ &= 41 + 40 + 39 + \cdots + 3 + 2 + 1 \\ &= 21 \times 41 = 861. \end{aligned}$$

28. (Also I28)
Let $10a + b$ be a number with at most two digits.

The equation $10a + b = 19(a + b)$ cannot hold unless $a = b = 0$. So all lucky numbers have at least three digits.

Suppose a lucky number has m digits for some $m \geq 4$. Then its digit sum is at most $9m$ while the number is at least 10^{m-1}. Hence $171m \geq 10^{m-1}$.

For $m = 4$, $684 \geq 1000$ is false, so there are no lucky numbers with 4 digits.

For $m \geq 5$, the situation is worse. Hence all lucky numbers have exactly three digits. Suppose the number is abc.

Then $100a + 10b + c = 19a + 19b + 19c$, we have $81a = 9b + 18c$ or $9a = b + 2c$.

For $a = 1$, we have $(b, c) = (1, 4)$, (3,3), (5,2), (7,1) and (9,0).
For $a = 2$, we have $(b, c) = (0, 9)$, (2,8), (4,7), (6,6) and (8,5).
For $a = 3$, we have $(b, c) = (9, 9)$, and there are no other solutions.

Hence there are exactly 11 lucky numbers, namely, 114, 133, 152, 171, 190, 209, 228, 247, 266, 285 and 399.

29. (Also I30)
The digits in base 10 which can be read upside down are 0, 1, 2, 5, 6, 8 and 9.

Then, writing numbers which can be read upside down is like writing numbers in base 7 using just those digits.

Writing 2007 in base is $5 \times 7^3 + 5 \times 7^2 + 6 \times 7 + 5 = 5565$.

But, in this pseudo base 7, the 5 is replaced by 8 and the 6 by 9.

So, 5565 is written as 8898 and is the 2007th number to be read upside down. The last three digits are 898.

30. (Alternative 1)
Given

$$\begin{aligned} x + y &= 3(z + u) && (1) \\ x + z &= 4(y + u) && (2) \\ x + u &= 5(y + z) && (3) \end{aligned}$$

Rewrite as

$$\begin{aligned} x + y &= 3z + 3u && (4) \\ x - 4y &= -z + 4u && (5) \\ x - 5y &= 5z - u && (6) \end{aligned}$$

Then, (4)-(5), (5)-(6) gives

$$\begin{aligned} 5y &= 4z - u && (7) \\ y &= -6z + 5u && (8) \end{aligned}$$

Hence

$$\begin{aligned} 5(-6z + 5u) &= 4z - u \\ 26u &= 34z \\ 13u &= 17z. \end{aligned}$$

If we let $u = 17$ and $z = 13$, the smallest possible values of z and u, then, from (8) we find $y = -78 + 85 = 7$ and from (4) we find $x = 39 + 51 - 7 = 83$. So 83 is the smallest possible value of x.

(Alternative 2)
Given

$$\begin{aligned} x+y &= 3(z+u) \quad (1)\\ x+z &= 4(y+u) \quad (2)\\ x+u &= 5(y+z) \quad (3) \end{aligned}$$

From (1) it follows that $x+y+z+u = 4(z+u)$.
Similarly, from (2) and (3) we have $x+y+z+u = 5(y+u)$
and $x+y+z+u = 6(y+z)$.

Thus if $S = x+y+z+u$, $4|S$, $5|S$ and $6|S$ so that $60|S$. Setting $S = 4\cdot5\cdot6 = 120$ we obtain

$$\begin{aligned} x+y = 3(z+u) &= \frac{3}{4}S = 90\\ x+z = 4(y+u) &= \frac{4}{5}S = 96\\ x+u = 5(y+z) &= \frac{5}{6}S = 100 \end{aligned}$$

Furthermore,

$$x = \frac{(x+y)+(x+z)+(x+u)-(x+y+z+u)}{2}$$

$$= \frac{90+96+100-120}{2} = \frac{166}{2} = 83.$$

If we set $S = 60 = l.c.m[4,5,6]$ we do not obtain an integer value for x.

Comment
A variation of a problem from Iamblichus of Chalcis (c.326).

* * *