Chapter 4 Linear Inequalities in One Unknown

4.1 Linear Inequalities

Look at the following expressions:

$$-7 < -5,\ 3 + 4 > 1 + 4,\ 2x < 6,\ a + 2 > a + 1,$$
$$5+3 \neq 12-5,\ \ a \neq 0.$$

These expressions contain some symbols that we have learnt before, such as '<', '>' and '≠'. These symbols are called inequality signs. Among these expressions, some only represent the left hand side is not equal to the right hand side; while some represent not only that the two sides are unequal, but also that one side has a greater value. These expressions representing unequal relationship are called **inequalities**.

'<' and '>', which represent 'is less than' and 'is greater than' repectively, are inequality signs denoting smaller and larger relations.

Practice

1. (*Mental*) Fill in the blanks with '<' or '>':

(1) -5 ___ -3; (2) -2 ___ 0;

(3) 9 ___ -14; (4) 3.254 ___ $3\frac{1}{4}$;

(5) $\frac{2}{7}$ ___ $\frac{1}{3}$; (6) $-\frac{1}{2}$ ___ $-\frac{1}{3}$.

2. (*Mental*) Fill in the blanks with '<' or '>':

(1) $7+3$ ___ $4+3$; (2) $7-3$ ___ $4-3$;

(3) 7×3 ___ 4×3; (4) $7\times(-3)$ ___ $4\times(-3)$.

Practice

3. Express the following in terms of an inequality:

(1) *a* is a positive number; (2) *a* is a negative number;

(3) *x* is not equal to 1; (4) $m + n$ is a positive number;

(5) 4 times *x* is greater than 7;

(6) Sum of *b* and 6 is less than 5.

Next we will investigate the basic properties of inequalities.

Look at the inequality

$$7 > 4.$$

First we look at the answer of Question 2 in the above exerecise and change 3 to 5. Then we do the following tests:

1. After adding (or subtracting) 5 on both sides, what is the result? Does the inequality sign change?

$$7+5 \ \_\_\_\ 4+5,\ \ 7-5 \ \_\_\_\ 4-5.$$

2. After multiplying 5 on both sides, what is the result? Does the inequality sign change?

$$7\times 5 \ \_\_\_\ 4\times 5.$$

3. After multiplying -5 on both sides, what is the result? Does the inequality sign change?

$$7\times(-5) \ \_\_\_\ 4\times(-5).$$

We observe that: inequality sign remains unchanged under the first and second situation. Under the third situation, after multiplying both sidesby a negative number, the inequality sign reverses.

When trying with a new inequality, say $-2 < 6$, the same behaviour is experienced.

In general, inequalities have the following basic properties:

Basic Property 1. When both sides of an inequality is increased or decreased by the same number, the inequality sign remains unchanged.

This means that if $a<b$, then $a+c<b+c$ (or $a+c<b+c$); if $a>b$, then $a+c>b+c$ (or $a+c>b+c$).

Basic Property 2: **When both sides of an inequality is multiplied (or divided) by the same positive number, the inequality sign remains unchanged.**

This means that if $a<b$, and $c>0$, then $ac<bc$ (or $\frac{a}{c}<\frac{b}{c}$); if $a>b$, and $c>0$, then $ac>bc$ (or $\frac{a}{c}>\frac{b}{c}$).

Basic Property 3: **When both sides of an inequality is multiplied (or divided) by the same negative number, the inequality sign is reversed.**

This means that if $a<b$, and $c<0$, then $ac>bc$ (or $\frac{a}{c}>\frac{b}{c}$); if $a>b$, and $c<0$, then $ac<bc$ (or $\frac{a}{c}<\frac{b}{c}$).

Line for thought: what will happen if we multiply both sides of the inequality by zero?

【Example 1】 Write down the result of the following inequality after applying the adjustment:

(1) $5<9$, adding -2 to both sides;
(2) $9>5$, subtracting 10 from both sides;
(3) $-5<3$, multiplying both sides by 4;
(4) $14>-8$, dividing both sides by -2;

Solution (1) According to Basic Property 1, inequality sign remains unchanged after adding -2 to both sides. So

$$5+(-2)<9+(-2),$$

That is $3<7$;

(2) According to Basic Property 1, we have

$$9-10>5-10,$$

That is $-1>-5$;

(3) According to Basic Property 2, we have

$$-5\times 4<3\times 4,$$

That is $-20<12$;

(4) According to Basic Property 3, we have

$$14\div(-2)<-8\div(-2),$$

That is $-7<4$.

【Example 2】 Let $a>b$, connect the following expressions with correct inequality sign:

(1) $a-3$ and $b-3$;
(2) $2a$ and $2b$;
(3) $-a$ and $-b$.

Solution (1) Since $a>b$, subtracting 3 from both sides under Basic Property 1, we have

$$a-3>b-3;$$

(2) Since $a>b$, multiplying 2 to both sides under Basic Property 2, we have

$$2a>2b;$$

(3) Since $a>b$, multiplying -1 to both sides under Basic Property 3, we have

$$-a<-b.$$

Practice

1. Write down the result of the inequality after applying the adjustment:
   (1) $-7<8$, adding 9 to both sides;
   $-7<8$, adding -9 to both sides.
   (2) $5>-2$, subtracting 6 from both sides;
   $5>-2$, subtracting -6 from both sides.
   (3) $-3>-4$, multiplying both sides by 7;
   $-3>-4$, multiplying both sides by -7.
   (4) $-8<0$, dividing both sides by 8;
   $-8<0$, dividing both sides by -8.

Practice

2. Let $a < b$. Connect the following expressions with correct inequality signs:
   (1) $a+5$ and $b+5$; (2) $3a$ and $3b$;
   (3) $-5a$ and $-5b$; (4) $\frac{a}{3}$ and $\frac{b}{3}$.

4.2 Solving Inequalities

Consider the inequality $2x < 6$. This is an inequality involving one unknown. The inequality is valid when x is substituted the value of 2 ; the inequality however becomes invalid when x is substituted the value of 3. Similar to solving equations, we say that 2 is a solution to the inequality $2x < 6$ but 3 is not.

We find out that 1, 0, -2.5 and -4, etc are also solutions to the inequality $2x < 6$. Indeed, substituting any number smaller than 3 for x still makes the inequality valid. On the contrary, substituting any number greater than or equal to 3 does not satisfy the inequality $2x < 6$. So, we know that the inequality $2x < 6$ has infinitely many solutions for x less than 3.

We say that all the solutions of the inequality $2x < 6$ form a set called the solution set of the inequality $2x < 6$. In general, all the solutions of an inequality constitute a **set** called the **solution set of the inequality**.

The solution set of the inequality $2x < 6$ is $x < 3$.

The process of finding the set of solutions of an inequality is called **solving the inequality**.

The solution set of an inequality can be directly represented on the number line. For example, if the solution set of an inequality is $x < 3$, it can be represented by all numbers left of number 3 on the number line (Diagram 4-1). The hollow circle mark at number 3 means that the number 3 itself is not contained in the solution set.

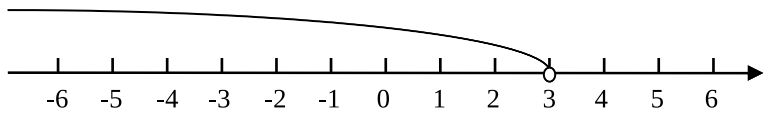

Diagram 4-1

If the solution set of an inequality is $x \geq -2$ (symbol '$\geq$' means 'greater than or equal to' or equivalently 'not less than'; similarly, symbol '$\leq$' means 'less than or equal to' or equivalently 'not greater than'), then it can be represented by all numbers right of number -2 on the number line (Diagram 4-2). The solid circle mark at number -2 means that the number -2 itself is contained in the solution set.

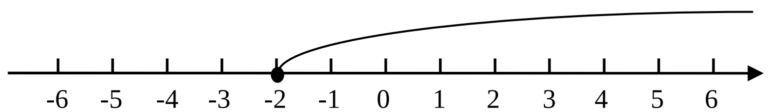

Diagram 4-2

Practice

1. Write down the inequality according to the following quantitative relatioship:
   (1) Three times of x is greater than 1;
   (2) Sum of x and 5 is negative;
   (3) Difference between y and 1 is positive;
   (4) Halve of x is not greater than 10.
2. Find the solution set of the following inequality and compare it with that of inequality $2x < 6$:
   (1) $2x+1 < 7$; (2) $4x < 12$.
3. Express the following solution set on the number line:
   (1) $x > 5$; (2) $x \geq 0$;
   (3) $x \leq 3$; (4) $x < -2\frac{1}{2}$.

4.3 Equivalent Inequalities

We know that the solution set of inequality $2x < 6$ is $x < 3$. Also, from the Question 2 of the above exerecise, we find out that the solution set of inequality $4x < 12$ is also $x < 3$. So the solution set of inequality $2x < 6$ and the solution set of inequality $4x < 12$ are the same.

Generally, if two inequalities have the same solution set, then the two inequalities are called **Equivalent inequalities**.

Therefore, inequalities $2x < 6$ and $4x < 12$ are equivalent inequalities. From the answer of Question 2 of the above exerecise, inequalities $2x < 6$ and $2x+1 < 7$ (that is $2x+1 < 6+1$) are also equivalent inequalities.

Regarding equivalent inequalities, there are three axioms.

Equivalent Inequality Axiom 1: If both sides of an inequality are increased (or decreased) by the same number or by the same integral expression, then the resulting inequality is an equivalent inequality to the original inequality;

Equivalent Inequality Axiom 2: If both sides of an inequality are multiplied (or divided) by the same positive number, then the resulting inequality is an equivalent inequality to the original inequality;

Equivalent Inequality Axiom 3: If both sides of an inequality are multiplied (or divided) by the same negative number, the resulting inequality with its inequality sign reversed is an equivalent inequality to the original inequality.

【**Example**】 Why are the following pair of inequalities equivalent inequalities?

(1) $21x < 14x + 8$ and $7x < 8$;
(2) $-5 + x \le -4$ and $x \le 1$;
(3) $-16x \ge -144$ and $x \le 9$.

Solution (1) If $14x$ is deducted from both sides of the inequality $21x < 14x + 8$, the resulting inequality $7x < 8$ is according to Equivalent Inequality Axiom 1 an equivalent inequality to the original indequality;

(2) If 5 is added to both sides of the inequality $-5 + x \le -4$, the resulting inequalitywe $x \le 1$ is according to Equivalent Inequality Axiom 1 an equivalent inequality to the original indequality;

(3) If both sides of the inequality $-16x \ge -144$ are divided by the negative number -16, the resulting inequaity $x \le 9$ with its inequality sign reversed is according to the Equaivalent Inequality Axiom 3 an equivalent inequality to the original inequality.

Observed in part (1) and part (2) of the above example, **after changing the sign of a term and then moving it to the other side of the inequality, the resulting inequality is an equivalent inequality to the original inequality**. It means that the mechanism of transposing terms used in solving equations is also applicable to solving inequalities.

NOTE: In applying Equivalent Inequality Axiom 3, always remember to reverse the inequality sign.

Practice

Why are the following pairs of inequalities equivalent inequalities?

(1) $3x \le 9$ and $x \le 3$;
(2) $2x - 7 < 6x$ and $-7 < 4x$;
(3) $8 + 3.5x \le 4.5$ and $3.5x \le -3.5$;
(4) $-4x < 448$ and $x > -112$;

4.4 Linear Inequalities in One Unknown and Their Solving Method.

Consider the inequality below:

$$2x < 6,\quad 4x - 7 > 3,\quad \frac{2y-1}{3} - y < 0.$$

This type of inequality which involves one unknown with degree 1 is called **a linear inequality in one unknown**.

Solving linear inequality in one unknown means finding the solution set of the inequality. The steps in solving linear inequations in one unknown are similar to those in solving linear equation in one unknown. However, it should be noted that when both sides of the inequality are multiplied (or dividied) by the same negative number, the resulting inequality will have the inequality sign reversed.

【**Example 1**】Solve the inequality $3(1-x)<2(x+9)$ and express the solution set on the number line.

Solution After removing brackets, we have

$$3-3x<2x+18.$$

Transposing terms, we have

$$-3x-2x<18-3.$$

Collecting like terms, we have

$$-5x<15.$$

Dividing both sides by -5, we have

$$x>-3.$$

The solution set is expressed on the number line as shown below (Diagram 4-3):

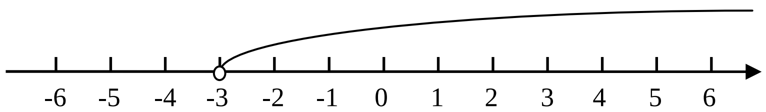

Diagram 4-3

【Remark】The hollow circle at the number -3 means that -3 is not contained in the solution set.

【**Example 2**】Solve the inequality $\dfrac{2+x}{2}\geq\dfrac{2x-1}{3}$ and express the solution on a number line.

Solution Removing the denominators (by multiplying both sides by 6), we have

$$3(2+x)\geq 2(2x-1).$$

Removing brackets, we have

$$6+3x\geq 4x-2.$$

Transposing terms, we have

$$3x-4x\geq -2-6.$$

Collecting like terms, we have

$$-x\geq -8.$$

Dividing both sides by -1, we have

$$x\leq 8.$$

The solution set is represented on the number line as shown below (Diagram 4-4):

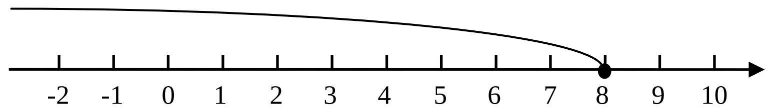

Diagram 4-4

【Remark】The solid circle means that 8 is contained in the solution set.

In solving example 2, we have $6+3x\geq 4x-2$ after removing brackets. If we move all the terms involving x to the right hand side of the inequality, we have

$$6+2\geq 4x-3x.$$

Collecting like terms, we have

$$8\geq x.$$

That is

$$x\leq 8.$$

Both solving methods are correct and the latter one is simpler.

【**Example 3**】What value of x should be taken if the value of the inequality $2x-5$ is:

(1) Greater than 0? (2) Not greater than 0?

Analysis:The question indeed asks what value of x should be taken such that the inequality

$$2x-5>0$$

becomes valid. Similarly, to find the solution set of $2x-5\leq 0$, we have to find the value of x such that $2x-5\leq 0$.

Solution (1) According to the question,

$$2x-5>0.$$

Solving the inquality to obtain the solution set, we have

$$2x>5$$

$$x>\frac{5}{2}$$

So when the value of x is greater than $\frac{5}{2}$, the value of $2x-5$ is greater than 0.

(2) According to the question,

$$2x-5\le 0.$$

Solving the inequality to obtain the solution set, we have

$$2x\le 5$$

$$x\le\frac{5}{2}$$

So when the value of x is not greater than $\frac{5}{2}$, the value of $2x-5$ is not greater than 0.

【Example 4】Find the positive integral solutions of the inequality $3x-10\le 0$.

Solution Solving $3x-10\le 0$, we have

$$x\le 3\frac{1}{3}.$$

Since positive integers 1, 2 and 3 are not greater than $3\frac{1}{3}$, so the solution set contains 1, 2 and 3.

Practice

1. Solve the following inequality and express the solution by using number line:
   (1) $x+3>2$; (2) $-2x<10$;
   (3) $3x+1<2x-5$; (4) $2-5x\ge 8-2x$;
   (5) $\frac{1}{2}(3-x)\ge 3$; (6) $1+3x\ge 5-\frac{x-2}{2}$.
2. What value of x should be taken so that the value of expression $3x+7$ is:
   (1) not less than 1; (2) not greater than 1.
3. Find the non-negative integral solution of the inequality $10(x+4)+x\le 84$.

Exercise 15

1. Write down the resulting inequality after applying the adjustment:
   (1) $5>-4$, adding 8 to both sides;
   (2) $1<3$, subtracting 4 from both sides;
   (3) $-3<-2$, multiplying both sides by 2;
   (4) $-14<20$, dividing both sides by 2;
   (5) $-4<-1$, dividing both sides by -3;
   (6) $-8<-4$, dividing both sides by -4.
2. Given that $a<b$, connect the following pairs of expressions by a correct inequality sign:
   (1) $a+1$ and $b+1$; (2) $a-3$ and $b-3$;
   (3) $-3a$ and $-3b$; (4) $\frac{a}{4}$ and $\frac{b}{4}$;
   (5) $-\frac{a}{7}$ and $-\frac{b}{7}$; (6) $a-b$ and 0.

3. According to the following quantitative relationship, write down the inequality:
   (1) $\frac{2}{3}$ of x minus 5 is less than 1;
   (2) Sum of x and 6 is not less than 9;
   (3) Sum of 8 and two times y is positive;
   (4) Three times a minus 7 is negative.

4. Express the solution sets of the inequality on the number line:
   (1) $x > 3$; (2) $x \geq -2$; (3) $x \leq 4$; (4) $x < 0$.

5. Why are the following pairs of inequalities equivalent inequalities?
   (1) $\frac{1}{2} > 2x$ and $1 > 4x$; (2) $4x - 2 \geq 6$ and $4x \geq 8$;
   (3) $3.14x < 0$ and $x < 0$; (4) $4 \leq -\frac{5}{17}x$ and $-68 \geq 5x$;
   (5) $-\frac{22}{7}x < 0$ and $x > 0$; (6) $-\frac{x}{4} > -\frac{2x}{3}$ and $3x < 8x$.

6. State which equivalent inequality axiom explains the transformation:
   (1) If $x + 2 > 7$, then $x > 7 - 2$;
   (2) If $3x > 1 - 2x$, then $3x + 2x > 1$;
   (3) If $2x < -5$, then $x < -\frac{5}{2}$;
   (4) If $-\frac{x}{2} < 3$, then $x > -6$.

7. Fill in the blank with correct '<' or '>' sign to cause the inequality on the left side and the inequality on the right side to become equivalent inequaities. State which equivalent inequality axiom explains the transformation.
   (1) If $-a < 5$, then a ___ -5;
   (2) If $3a > 6$, then a ___ 2.

8. Solve the following inequality and express the solution set on the number line:
   (1) $5x > -10$; (2) $-3x < -12$;
   (3) $\frac{x}{2} \geq 3$; (4) $-\frac{3x}{5} < -3$;
   (5) $8x - 1 \geq 6x + 5$; (6) $3x - 5 < 1 + 5x$;
   (7) $3(2x + 5) > 2(4x + 3)$; (8) $10 - 4(x - 3) \leq 2(x - 1)$;
   (9) $\frac{x-3}{2} > \frac{x+6}{5}$; (10) $\frac{2(4x-3)}{3} \geq \frac{5(5x+12)}{6}$.

9. Solve the following inequalities:
   (1) $\frac{x+5}{2} - \frac{3x+2}{2} \geq 2x$; (2) $\frac{y+1}{3} - \frac{y-1}{2} \geq \frac{y-1}{6}$;
   (3) $2 + \frac{3(x+1)}{8} > 3 - \frac{x-1}{4}$; (4) $\frac{3x-2}{3} - \frac{9-2x}{3} \leq \frac{5x+1}{2}$.

10. Without computing the value of the product of the following pair of numbers, state whether the product is greater than, less than or equal to 0.
   (1) 3 and 2; (2) $-\frac{1}{5}$ and $-\frac{1}{2}$;
   (3) -0.4 and 0.7; (4) 1.5 and -6.

11. Fill in the blank with correct '<' or '>' sign:
   (1) When $a > 0$ and $b > 0$, then ab ___ 0;
   (2) When $a < 0$ and $b > 0$, then ab ___ 0;
   (3) When $a < 0$ and $b < 0$, then ab ___ 0;
   (4) When $a > 0$ and $b < 0$, then ab ___ 0.

12. What value of x would cause the value of the expression $4x + 8$ to become:
   (1) a positive number?
   (2) a negative numbere?
   (3) zero?

Chapter Summary

I. This chapter mainly teaches inequalities, their basic properties, solution sets of inequalities, Equivalent Equation Axioms, linear inequalitis in one unknown and their solving methods.

II. In our daily life, when we compare different quanitites of the same kind (such as lengths and time), we may express the relationshnip as equal or unequal. Equal relationship can be expressed by an equation and unequal relationship can be expressed by an inequality. Given any two numbers *a* and *b*, it is only possible that one and only one of the following three relationships, (i) $a < b$, (ii) $a = b$, and (iii) $a > b$, is true.

III. The following table compares the difference between Equivalent Axioms of equations and inequalities, and the difference btween solving procedure of linear equations in one unknown and linear inequalities in one unknown:

<table>
<tr><th></th><th>Equations</th><th>Inequalities</th></tr>
<tr><td rowspan="3">Equivalent Axioms</td><td>If both sides of an equattion are increased (or decreased) by the same number or by the same integral expression, then the resulting equation is an equivalent equation to the original equation.</td><td>If both sides of an inequality are increased (or decreased) by the same number or by the same integral expression, then the resulting inequality is an equivalent inequality to the original inequality.</td></tr>
<tr><td rowspan="2">If both sides of an equation are multiplied (or divided) by the same number (whether positive or negative), then the resulting equation is an equivalent equation to the original equation.</td><td>If both sides of an inequality are multiplied (or divided) by the same positive number, then the resulting inequality is an equivalent inequality to the original inequality.</td></tr>
<tr><td>If both sides of an inequality are multiplied (or divided) by the same negative number, then the resulting inequality with its inequality sign reversed is an equivalent inequality to the original inequality</td></tr>
</table>

	Equations	Inequalities
Solving Procedure	Solving linear equations in one unknown: 1. Removing denominators; 2. Removing brackets; 3. Transposing terms; 4. Collecting like terms; 5. Dividing both sides of equation by the coefficient of the unknown.	Solving linear inequalities in one unknown: 1. Removing denominators; 2. Removing brackets; 3. Transposing terms; 4. Collecting like terms; 5. Dividing both sides of the inequalities by the coefficient of the unknown. In Steps 1 and 5, if both sides of the equation are multiplied or divided by a negative number, then the inequality sign is reversed.
Number of Solutions	A linear equation in one unknown has only one solution.	A linear inequality in one unknown has infinitely many solutions.

Revision Exercise 4

1. What are the basic properties of inequalities?
2. Explain, with examples, what is meant by the solution set of an inequality and what is meant by equivalent inequalities.
3. What are the Equivalent Inequality Axioms? What are the similarities and differences between Equivalent Equation Axioms and Equivalent Inequality Axioms?

4. Solve the following inequality and express the solution on the number line:

(1) $2(x-3)>4$; (2) $2x-3\le 5(x-3)$;

(3) $\frac{1}{5}(x-2)\le x-\frac{2}{5}$; (4) $\frac{x}{3}-\frac{x-1}{2}<1$.

5. Is the following solution correct? Why?

(1) $-x=6$, multiplying both sides by -1, then we have $x=-6$;

(2) $-x>6$, multiplying both sides by -1, then we have $x>-6$;

(3) $-x\le 6$, multiplying both sides by -1, then we have $x\le -6$.

6. Is the following solution correct? Why?

(1)
$$\begin{aligned} 7x+5&>8x+6\\ 7x-8x&>6-5\\ -x&>1\\ x&>-1 \end{aligned}$$

(2)
$$\begin{aligned} 6x-3&<4x-4\\ 6x-4x&<-4+3\\ 2x&<-1\\ x&>\frac{1}{2} \end{aligned}$$

7. Solve the following inequality:

(1) $2(3x-1)-3(4x+5)>x-4(x-7)$;

(2) $3[x-2(x-1)]\le 4x$;

(3) $\frac{1}{2}(3x-1)+\frac{x}{5}<7x+10.1$;

(4) $5+\frac{x}{-3}\ge 3\frac{1}{2}-\frac{4x+1}{8}$.

8. What value of a should be taken such that the value of the expression $3-2a$ is:

(1) greater than 1? (2) equal to 1? (3) less than 1?

9. What value of y should be taken such that the value of the expression $\frac{y}{3}-3$ is:

(1) greater than $\frac{y}{2}-3$? (2) less than $\frac{y}{2}-3$?

10. (1) Is it correct that $5a>4a$?

(2) Is it correct that $\frac{a}{3}$ is less than a?

(3) Comparing a and $-a$, which one is greater?

11. Adding 5 to two times of an unknown is not greater than subtracting 4 from three times of the unknown. Find the solution set of this unknown.

12. Find the number of positive integral solutions of the inequality $64-3x>4$.

13. The sum of three consecutive natural numbers is less than 15. How many such groups of consecutive natural numbers are there?

14. (1) If a positive number is greater than another positive number, then which number has a greater absolute value? Explain your answer with an example.

(2) If a negative number is greater than another negative number, then which number has a greater absolute value? Explain your answer with an example.

(3) If a number is greater than another number, then which number has a greater opposite number? Explain your answer with an example.

*15. Fill in the blanks with the correct '<' or '>' sign:

(1) If $a<b$, then $a-b$ ___ 0;

(2) If $a=b$, then $a-b$ ___ 0;

(3) If $a>b$, then $a-b$ ___ 0.

*16. Fill in the blanks with the correct '<' or '>' sign:

(1) If $a-b<0$, then a ___ b;

(2) If $a-b=0$, then a ___ b;

(3) If $a-b>0$, then a ___ b.

*17. Fill in the blanks with the correct '<' or '>' sig:

(1) If $a>0$ and b ___, then $ab>0$;

(2) If $a>0$ and b ___, then $ab<0$.

*18. (1) Is it always true that $a+b$ is greater than a? Why?
(Hint: Consider separately for the three cases: $b>0$, $b=0$, $b<0$)

(2) Is it always true that $a-b$ is greater than a? Why?

(The chapter is translated by courtesy of Mr. LAI Kit Ming, and reviewed by courtesy of Mr. SIN Wing Sang, Edward)