Chapter 12 Indicies

We have learnt about power of base with positive integral indicies, and understand they obey the following rules of operations:

(1) $a^m \bullet a^n = a^{m+n}$

(2) $a^m \div a^n = a^{m-n}$ $(a \neq 0,\ m > n)$

(3) $(a^m)^n = a^{mn}$

(4) $(ab)^n = a^n \bullet b^n$

(5) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $(b \neq 0)$

Now, we would like to explore about power of base with zero index and with negative integral indices.

12.1 Power of Base with Zero Index and with Negative Integral Indices

We know from basic operation rule (2) that for division of powers of the same base, the result is a power of the same base with index equal to the diffrence of the indices of the two powers:

$$a^m \div a^n = a^{m-n}.$$

To avod division by zero, we require $a \neq 0$; to ensure that the result is a power with positive index, we require $m > n$. However in actual calculation, we may encounter situation that (i) $m = n$ or (ii) $m < n$. We shall explore these two special situations.

1. Power with Zero Index

As we know, when both the dividend and the divisor are powers of the same base and with the same index, the quotient equals 1. For example,

$$5^2 \div 5^2 = 1$$

$$a^3 \div a^3 = 1 \quad (a \neq 0)$$

On the other hand, if we apply operation rule (2) as above, the quotient is a power of the same base with index of zero after subtraction. That is, from the division, we get

$$5^2 \div 5^2 = 5^{2-2} = 5^0$$

$$a^3 \div a^3 = a^{3-3} = a^0 \quad (a \neq 0)$$

Here we encounter situation of power with zero index.

In order to extend operation rule (2) to apply to cases where the indices of the dividend and the divisor are equal, we shall define power with zero index to have the following meaning

$$\boxed{\boldsymbol{a^0 = 1 \ \ (a \neq 0)}}$$

That is to say, the value of the power of any base with zero index is equal to 1.

With the above definition, we can write the division calculation as follows:

$$5^2 \div 5^2 = 5^{2-2} = 5^0 = 1$$

$$a^3 \div a^3 = a^{3-3} = a^0 = 1 \quad (a \neq 0)$$

Care should be taken that the power of zero (as base) with zero index does not have meaning.

2. Power with Negative Integral Indices

When dividing power with the same base, if the index of the power of the dividend is less than the index of the power of the divisor, we can use normal cancellation method to reduce the fraction. For example

$$5^2 \div 5^6 = \frac{5^2}{5^6} = \frac{5^2}{5^2 \bullet 5^4} = \frac{1}{5^4}$$

$$a^3 \div a^5 = \frac{a^3}{a^5} = \frac{a^3}{a^3 \bullet a^2} = \frac{1}{a^2} \quad (a \neq 0)$$

We observe that, when dividing power with the same base, if the index of the power of the dividend is less than the power of the divisor by *p*, the quotient is a fraction with the numerator equals 1 and the denominator equals the power of the same base with index *p*.

On the other hand, if we divide powers following operation rule (2) to subtract the indicies, we get

$$5^2 \div 5^6 = 5^{2-6} = 5^{-4}$$

$$a^3 \div a^5 = a^{3-5} = a^{-2} \quad (a \neq 0)$$

We obtain an answer in the form of a power with the same base but with a negatibve integral index.

To enable operation rule (2) to apply even though the index of power of the dividend is less thatn the index of power of the divisor, we define the power of a base with negative index to have the following meaning

$$\boxed{a^{-p} = \frac{1}{a^p} \quad (a \neq 0\text{, } p \text{ is a positive integer})}$$

That is to say, any power of a base with a negative index $-p$ (where p is a positive integer), the value is equal to the reciprocal of power with index p.

With this definiation, we can perform the above calculation as follows:

$$5^2 \div 5^6 = 5^{2-6} = 5^{-4} = \frac{1}{5^4}$$

$$a^3 \div a^5 = a^{3-5} = a^{-2} = \frac{1}{a^2} \quad (a \neq 0)$$

Care should be taken that the power of base zero with negative integral index does not have meaning.

Having defined the meaning of power with indices zero and negative integral values, we can extend the operation rules for powers with positive integers to powers with any integers (including zero and negative integers). For example:

$$a^3 \bullet a^0 = a^{3+0} = a^3 \qquad (a \neq 0)$$

$$a^{-3} \bullet a^2 = a^{-3+2} = a^{-1} = \frac{1}{a} \qquad (a \neq 0)$$

$$(a^{-3})^2 = a^{-3\times 2} = a^{-6} = \frac{1}{a^6} \qquad (a \neq 0)$$

In this chapter, when the index is zero or negative integers, we can take it for granted that the base is not zero unless it is explicitly mentioned.

【Example 1】 Compute

$$10^{-3}, \quad (-3)^{-2}, \quad \left(\frac{1}{2}\right)^{-3}, \quad 5^0 \times (-2)^{-1}.$$

Solution $10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$; $\qquad (-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{9}$;

$$\left(\frac{1}{2}\right)^{-3} = \frac{1}{\left(\frac{1}{2}\right)^3} = 8; \qquad 5^0 \times (-2)^{-1} = 1 \times \frac{1}{-2} = -\frac{1}{2}.$$

【Example 2】 Express the following in decimal:

$$10^{-5}, \quad 7\times 10^{-6}, \quad 3.6\times 10^{-8}.$$

Solution $10^{-5} = \frac{1}{10^5} = 0.00001$;

$$7\times 10^{-6} = 7\times\frac{1}{10^6} = 7\times 0.000001 = 0.000007;$$

$$3.6\times 10^{-8} = 3.6\times\frac{1}{10^8} = 3.6\times 0.00000001 = 0.000000036.$$

Practice

1. (*Mental*) What is the answer to the following question?

   (1) $3a^2b + 2a^2b$; (2) $3a^2b \bullet 2a^2b$;

   (3) $(3ab^2)^2$; (4) $\left(-\frac{2b}{a^2}\right)^3$;

   (5) $16a^4b^2 \div 12a^2b^2$; (6) $(a^2b^2)^3 \div a^2b$.

2. Compute:

$$3^0,\ 3^{-1},\ 10^{-4},\ (\sqrt{2})^0,\ 7^{-2},\ 1^{-10},\ (-2)^{-3},$$

$$\left(\frac{1}{2}\right)^{-4},\ (-0.1)^0,\ \left(-\frac{1}{2}\right)^{-3}.$$

Practice

3. Compute:

(1) $(-2)^3-(-1)^0$; (2) $2^{-2}+(-2)^{-5}$;

(3) $\left(\frac{1}{2}\right)^{-2}\div\left(\frac{1}{2}\right)^{0}$; (4) $\left(-\frac{1}{2}\right)^{-2}\times 2^{-1}$.

4. Express the following in decimal:

(1) 2×10^{-5}; (2) 3.1×10^{-7};

(3) 8.04×10^{-8}; (4) 1.205×10^{-2};

(5) 2.12×10^{-3}; (6) 2.12×10^{-2};

(7) 2.12×10^{-1}; (8) 2.12×10^{0}.

【**Example 3**】Compute $(-a)^{-5}$, $a^{-2}b^{-1}(-2a^3)$, $(-5a^3b^{-1})^{-2}$, and express the result in powers of positive indices.

Solution $(-a)^{-5}=\frac{1}{(-a)^5}=-\frac{1}{a^5}$;

$$a^{-2}b^{-1}(-2a^3)=-2a^{-2+3}b^{-1}=-2ab^{-1}=-\frac{2a}{b};$$

$$\begin{aligned}(-5a^3b^{-1})^{-2}&=(-5)^{1\times(-2)}a^{3\times(-2)}b^{(-1)\times(-2)}\\&=(-5)^{-2}a^{-6}b^{2}\\&=\frac{1}{(-5)^2}\times\frac{1}{a^6}\times b^2\\&=\frac{b^2}{25a^6}\end{aligned}$$

【**Example 4**】Compute: (1) $\frac{a^{-2}b^{-3}(-3a^{-1}b^2)}{6a^{-3}b^{-2}}$;

(2) $(x^{-2}+y^{-2})(x^{-2}-y^{-2})$;

(3) $\frac{a^{-1}+b^{-1}}{a^{-1}\bullet b^{-1}}$.

Solution (1)
$$\begin{aligned}\frac{a^{-2}b^{-3}(-3a^{-1}b^2)}{6a^{-3}b^{-2}}&=-\frac{3}{6}a^{-2+(-1)-(-3)}b^{-3+2-(-2)}\\&=-\frac{1}{2}a^0b\\&=-\frac{1}{2}b\end{aligned}$$

(2) $(x^{-2}+y^{-2})(x^{-2}-y^{-2})=(x^{-2})^2-(y^{-2})^2=x^{-4}-y^{-4}$;

(3) $\frac{a^{-1}+b^{-1}}{a^{-1}\bullet b^{-1}}=\frac{(a^{-1}+b^{-1})ab}{(a^{-1}\bullet b^{-1})ab}=\frac{b+a}{1}=a+b$.

Practice

1. Compute the following and express the result in powers of positive indicies:

(1) $\frac{ab}{c^{-2}}$; (2) $pq^{-1}r^{-1}$;

(3) $\frac{a(a+b)^{-1}}{a^{-2}b}$; (4) $\frac{5^{-1}}{2^{-3}}\frac{xy^{-2}}{ab^{-4}}$.

2. Express the following in format without any fraction:

(1) $\frac{1}{y^5}$; (2) $\frac{a^2}{b^3}$; (3) $\frac{m^2}{x^5y}$.

3. (*Mental*) If the following equation correct？ If not, how should it be corrected?

(1) $(-1)^0=-1$; (2) $(-1)^1=1$;

(3) $3a^{-2}=\frac{1}{3a^2}$; (4) $(-x)^5\div(-x)^3=-x^2$.

4. Compute the following and express all the powers with positive indicies:

(1) $3^{-5}\bullet 3^6$; (2) $7^{-9}\div 7^{-10}$; (3) $a^{-3}\bullet a^2$;

(4) $b^{-4}b^{-2}$; (5) $(a^{-3})^{-2}$; (6) $(x^{-3})^0$;

(7) $(xy)^{-2}$; (8) $\left(\frac{p}{q}\right)^{-2}$.

Practice

5. Compute:

(1) $(x^4y^{-3})\bullet(x^{-2}y^2)$; (2) $3a^{-2}b^{-3}\div 3^{-1}a^2b^{-3}$;

(3) $\left(\dfrac{3^{-5}\bullet 3^2}{3^{-3}}\right)^{-2}$; (4) $\dfrac{(x^{-1}+y^{-1})(x^{-1}-y^{-1})}{x^{-2}y^{-2}}$.

3. Scientific Notation

Division of powers of the same base, if the index of power of the dividend is less that the index of power of the divisor, we can simplify the fraction using cancellation. For example, the land area of the Earth is 510000000 km^2. It can be represented as 5.1×10^8 km^2. Now, our knowledge of indicies of powers has extended from powers with positive integer to powers with any integer, we can represent any number using power of 10. For example, the thickness of a page of a book is 0.000075 m

$$0.000075=7.5\times0.00001=7.5\times10^{-5}$$

In this manner, we can describe the thickness of a paper as 7.5×10^{-5} m.

This method of representing numbers using powers of 10 is commonly used in science study, therefore we call such respresentation Scientific Notation. The format of scientific notation is to transform the number to the form $\pm a\times10^n$, where n is an integer, a is a number greater than 1 and less than 10.

We shall look at two examples below.

【Example 5】 Use scientific notation to represent the following numbers: 1000000, -30000, 57000000, -849000, 21.23, 5.08.

Solution $1000000=1\times1000000=1\times10^6$

$-30000=-3\times10000=-3\times10^4$

$57000000=5.7\times10000000=5.7\times10^7$

$-847000=-8.49\times100000=-8.49\times10^5$

$21.23=2.123\times10=2.123\times10^1$

$5.08=5.08\times1=5.08\times10^0$

We can observe from Example 5 that, using scientific notation to represent a number with absolute value greater than 1 is in the format $\pm a\times10^n$, where n is a non-negative number, and n equals the number of digits of the number minus 1.

【Example 6】 Use scientific notation to represent the following numbers: 0.008, -0.000034, 0.0000000125

Solution $0.008=8\times0.001=8\times10^{-3}$

$-0.000034=-3.4\times0.00001=-3.4\times10^{-5}$

$0.0000000125=1.25\times0.00000001=1.25\times10^{-8}$

We observe from Example 6 that, using scientific notation to represent a number with absolute value less than 1 is in the format $\pm a\times10^n$, where n is a negative integer, the absolute value of n equals the number of zeros before the first non-zero digit of the number (the zero in the unit position before the decimal point is included in the count).

Using scientific notation to represent a number with many digits, it is more convenient to read, write, compute and store the number.

【Example 7】 The mass of the Earth is 5.98×10^{21} T. The mass of Jupiter is is 318 times that of the Earth. Find the mass of Jupiter in T (correct to 2 significant figures)?

Solution $5.98\times10^{21}\times318=1901.64\times10^{21}\approx1.9\times10^{24}$.

Answer: Mass of Jupiter is 1.9×10^{24} T.

Practice

1. Use scientific notation to represent the following numbers:
   10000, 800000, 56000000, 2030000000, 7400000.
2. What number is represented by the following scientific notation?
   1×10^7, 4×10^3, 8.5×10^6, 7.04×10^5, 3.96×10^4.
3. Use scientific notation to represent the following numbers:
   (1) 0.00007; (2) 0.0000043;
   (3) 0.00000000807; (4) 0.0000006002;
   (5) 0.301; (6) -0.004025.

Practice

4. Use scientific notation to represent the following numbers:

(1) 153.7;	(2) 347200000
(3) 0.0000003142;	(4) 0.00000002001
(5) -6;	(6) 30.5771;
(7) 0.513;	(8) 0.002074.

5. Write the number represented by the following scientific notation:

(1) -3.05×10^{-6};	(2) 7.03×10^{5};	(3) -3.73×10^{-1};
(4) 2.14×10^{6};	(5) 1×10^{-8};	(6) 1.381×10^{7};
(7) 7×10^{1};	(8) 2.818×10^{3}.	

12.2 Power with Fractional Indices

1. Surd

We have learnt in earlier chapters about surds of second order (or quadratic surds) and their characteristics. Now we would like to explore about surds of general order and their characteristics.

We know, when n is odd, a number, a (whether positive or negative), taken to the n^{th} root is represented by $\sqrt[n]{a}$; when n is even, a non-negative number a taken to the n^{th} root is represented by $\sqrt[n]{a}$. The expression $\sqrt[n]{a}$ is called a **surd**, n is the order of the surd, a is called the base or radicand. We know, when the order of the surd n equals 2, it is also called a quadratic surd (here the order of the surd 2 can be abbreviated. When n equals 3, 4, 5, $\cdots$, we called it the cubic surd, quartic surd, quantic surd, and so on (here the order of the root cannot be abbreviated). If n is odd, a can be any real number; if n is even, a can only be a non-negative number. For example, $\sqrt{5}$, $\sqrt[3]{-5}$, $\sqrt[4]{\frac{2}{3}}$, $\sqrt[5]{a}$, $\sqrt[6]{b^2+1}$, $\sqrt{(a-b)^2}$ are all surds, $5\sqrt[4]{x^2+y^2}$ is also a surd. Care should be taken that, in the domain of real numbers, if the base is a negative number, taking root of an even order does not have meaning.

According to the meaning of surds, we have

(1) $(\sqrt{5})^2 = 5$, $(\sqrt[3]{-2})^3 = -2$;

(2) $\sqrt[3]{(-2)^3} = -2$, $\sqrt[5]{2^5} = 2$;

(3) $\sqrt{2^2} = 2$, $\sqrt{(-2)^2} = |-2| = -(-2) = 2$,

$\sqrt[4]{(-3)^4} = |-3| = -(-3) = 3$.

In general, if the expression $\sqrt[n]{a}$ has meaning, then

> **(1)** $(\sqrt[n]{a})^n = a$**;**
>
> **(2) If n is odd,** $\sqrt[n]{a^n} = a$**;**
>
> **(3) If n is even,** $\sqrt[n]{a^n} = |a| = \begin{cases} a & (a \ge 0) \\ -a & (a < 0) \end{cases}$

While the even root of a negative number does not have meaning, the odd root of a negative number can be found by taking the negative sign out of the surd and taking the odd root of the remaining positive base (or radicand). For example $\sqrt[5]{-2} = -\sqrt[5]{2}$, therefore when we explore the characteristics of surds, we need only study their arithmetic characteristics. We abide by the rule, that unless it is explicitly stated, all the bases inside the surd must be positive numbers.

According to surd rule $(\sqrt[n]{a})^n = a$, if $a \ge 0$, we can compute as follows:

$$(\sqrt[8]{a^6})^8 = a^6$$

$$(\sqrt[4]{a^3})^8 = [(\sqrt[4]{a^3})^4]^2 = (a^3)^2 = a^6$$

Both $\sqrt[8]{a^6}$ and $\sqrt[4]{a^3}$ are the 8^{th} rool of a^6, and the root of the 8^{th} order of a^6 should be unique with only one answer. Therefore if $a \ge 0$, we have

$$\sqrt[8]{a^6} = \sqrt[4]{a^3}.$$

With the same argument, we can deduce the following:

$$\boxed{\sqrt[np]{a^{mp}} = \sqrt[n]{a^m} \quad (a \ge 0)}$$

and

$$\boxed{\sqrt[p]{a^{mp}} = a^m \quad (a \ge 0)}$$

Here m is a positive integer, both n, p are integers greater than 1.

That is to say, when the order of a surd is a non-negative number, if the order of the surd and the index of the base are both multiplied (or divided) by the same positive integer, the value of the surd remains unchanged. This is the **Basic Property of Surd**.

Care should be taken to note that, for this Basic Property of Surd to be valid, it is required that to satisfy the requirement that the base $a \ge 0$, otherwise the basic property does not hold. For example, $\sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$, $\sqrt[3]{-8} = -2$, therefore $\sqrt[6]{(-8)^2} \ne \sqrt[3]{-8}$.

Surds with the same order of root are called **surds with root of like order**; Surds with different orders of roots are called **surds with root of unlike orders**. Using the Basic Property of Surd, we can transform surds with root of unlike orders to become surds with root of like order.

【Example 1】 Transform $\sqrt{a}$, $\sqrt[3]{a^2 b}$, $\sqrt[6]{a}$ to become surds with root of like order.

Analysis: The orders of roots of the three surds are 2, 3, 6, of which the least common multiple is 6. Therefore they can be transformed to become surds with root of the 6^{th} order.

Solution $\sqrt{a} = \sqrt[6]{a^3}$;

$\sqrt[3]{a^2 b} = \sqrt[6]{(a^2 b)^2} = \sqrt[6]{a^4 b^2}$;

$\sqrt[6]{a} = \sqrt[6]{a}$.

【Example 2】 Transform $\sqrt[3]{-5}$, $\sqrt[4]{3}$ to surds with root of like order.

Solution $\sqrt[3]{-5} = -\sqrt[3]{5} = -\sqrt[12]{5^4} = -\sqrt[12]{625}$;

$\sqrt[4]{3} = \sqrt[12]{3^3} = \sqrt[12]{27}$.

【Example 3】 Simplify the following surd by offsetting the order of root of the surd and the index of the base:

(1) $\sqrt[5]{a^{10}}$; (2) $\sqrt[6]{(-3)^2 x^4}$.

Solution (1) $\sqrt[5]{a^{10}} = a^2$;

(2) $\sqrt[6]{(-3)^2 x^4} = \sqrt[6]{3^2 x^4} = \sqrt[6]{(3x^2)^2} = \sqrt[3]{3x^2}$.

【Example 4】 Evaluate $\sqrt[6]{8}$, correct to the nearest 0.001.

Solution $\sqrt[6]{8} = \sqrt[6]{2^3} = \sqrt{2} \approx 1.414$.

Practice

1. On the assumption that x is a real number, find the condition for the following expression to have meaning:
   $\sqrt{x}$, $\sqrt{-x}$, $\sqrt[3]{x}$, $\sqrt[3]{-x}$, $\sqrt[4]{1-x}$, $\sqrt[4]{x-1}$.
2. Compute:
   (1) $\sqrt{x^2 - 2x + 1}$ $(x > 1)$; (2) $\sqrt[4]{(x^2 - 2x + 1)^2}$ $(x < 1)$.
3. Transform the following surds to surds with root of like order:
   (1) $\sqrt{5}$, $\sqrt[4]{2}$; (2) $\sqrt[3]{6y^2}$, $\sqrt[5]{-y}$;
   (3) $\sqrt{2mn}$, $\sqrt[5]{-6m^2 n}$, $\sqrt[10]{5m}$;
   (4) $\sqrt{x+y}$, $\sqrt[4]{x^2 + y^2}$, $\sqrt[6]{(x+y)^5}$.
4. Simplify the following surd by offsetting the order of the root of the surd with the index of the base:
   (1) $\sqrt[4]{x^2}$; (2) $\sqrt[3]{y^9}$; (3) $\sqrt[12]{x^4 y^6}$;
   (4) $\sqrt[6]{(-5)^4 a^4 b^2}$; (5) $\sqrt[4]{16 x^8 y^{12}}$; (6) $\sqrt[12]{a^{4m} b^{8n}}$.

2. Powers with Fractional Indicies

Let us look at the following examples:

$$\sqrt{a^6} = a^3 = a^{\frac{6}{2}} \qquad (a > 0)$$

$$\sqrt[3]{x^{12}} = x^4 = x^{\frac{12}{3}} \qquad (x > 0)$$

That is to say, when the order of root of the surd can be exactly divided by the index of the base, the surd can be simplified to a power of the base.

To facilitate computation, even though the order of root of the surd cannot be exactly divided by the index of the base, we can still represent the surd by a power with fractional index. For example

$$\sqrt[3]{a^2} = a^{\frac{2}{3}}, \quad \sqrt{b} = b^{\frac{1}{2}}, \quad \sqrt[4]{c^5} = c^{\frac{5}{4}}.$$

We define a power with index of positive fraction as follows:

$$\boxed{\boldsymbol{a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad (a > 0,\ m,\ n \text{ are both positive integers, } n > 1)}}$$

That is to say, a positive number rasied to the fractional index of $\frac{m}{n}$ (m, n are both integers, $n > 1$) equals the root of the n^{th} order of a base which is the positive integer raised to the m^{th} order.

A positive number raised to a negative integral index takes the meaning equal to the reciprocal of the same positive number rasied to the corresponding postive integral index. In a like manner, a positive number raised to a negative fractional index takes the meaning equal to the reciprocal of the same positive number raised to the corresponding positive index. That means

$$\boxed{\boldsymbol{a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{\sqrt[n]{a^m}} \quad (a > 0,\ m,\ n \text{ are both positive integers, } n > 1)}}$$

That is to say, the power of a positive number raised to the index $-\frac{m}{n}$ (m, n are both positive integers, $n > 1$) equals to the reciprocal of the power of the same positive number raised to the index $\frac{m}{n}$.

Care should be taken to note that a power of zero raised to positive fractional index equal zero. A power of zero raised to a negative fractional index does not ave meaning.

In this chapter, when the index is a fraction and the base number is not explicitly specified, the base number must be a positive number.

After defining power of number with fractional index to have meaning, all the operation rules concerning power of number with integeral index are equally applicable to power of number with rational index. For example,

$$a^{\frac{2}{3}} \bullet a^{-\frac{1}{4}} = a^{\frac{2}{3}+\left(-\frac{1}{4}\right)} = a^{\frac{5}{12}}.$$

【Example 5】 Find the value of the following expression:

$$8^{\frac{2}{3}}, \quad 100^{-\frac{1}{2}}, \quad \left(\frac{16}{81}\right)^{-\frac{3}{4}}.$$

Solution $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$;

$$100^{-\frac{1}{2}} = (10^2)^{-\frac{1}{2}} = 10^{-1} = \frac{1}{10};$$

$$\left(\frac{16}{81}\right)^{-\frac{3}{4}} = \left(\frac{2^4}{3^4}\right)^{-\frac{3}{4}} = \frac{2^{-3}}{3^{-3}} = \frac{3^3}{2^3} = \frac{27}{8}.$$

【Example 6】 Simplify the following expression and express the answer in powers with positive integral indices only:

(1) $(2a^{\frac{2}{3}}b^{\frac{1}{2}})(-6a^{\frac{1}{2}}b^{\frac{1}{3}}) \div (-3a^{\frac{1}{6}}b^{\frac{5}{6}})$;

(2) $(p^{\frac{1}{4}}q^{-\frac{3}{8}})^8$; (3) $\sqrt[4]{\left(\frac{16m^{-4}}{81n^4}\right)^3}$.

Solution (1) $(2a^{\frac{2}{3}}b^{\frac{1}{2}})(-6a^{\frac{1}{2}}b^{\frac{1}{3}})\div(-3a^{\frac{1}{6}}b^{\frac{5}{6}}) = 4a^{\frac{2}{3}+\frac{1}{2}-\frac{1}{6}}b^{\frac{1}{2}+\frac{1}{3}-\frac{5}{6}}$

$$= 4ab^0$$

$$= 4a$$

(2) $(p^{\frac{1}{4}}q^{-\frac{3}{8}})^8 = (p^{\frac{1}{4}})^8(q^{-\frac{3}{8}})^8 = p^2q^{-3} = \dfrac{p^2}{q^3}$;

(3) $\sqrt[4]{\left(\dfrac{16m^{-4}}{81n^4}\right)^3} = \left(\dfrac{2^4m^{-4}}{3^4n^4}\right)^{\frac{3}{4}} = \dfrac{2^3m^{-3}}{3^3n^3} = \dfrac{8}{27m^3n^3}$.

Practice

1. Express the following in power form without any fraction:

$$\sqrt[3]{x^2}\ ,\ \frac{1}{\sqrt[3]{a}}\ ,\ \sqrt[4]{(a+b)^3}\ ,\ \sqrt[3]{m^2+n^2}\ ,\ \frac{\sqrt{x}}{\sqrt[3]{y^2}}\ .$$

2. Compute:

(1) $25^{\frac{1}{2}}$; (2) $\left(\dfrac{81}{25}\right)^{-\frac{1}{2}}$; (3) $27^{\frac{2}{3}}$;

(4) $10000^{\frac{1}{4}}$; (5) $4^{-\frac{1}{2}}$; (6) $\left(6\dfrac{1}{4}\right)^{\frac{3}{2}}$;

(7) $2^{-1}\times 64^{\frac{2}{3}}$; (8) $(0.2)^{-2}\times(0.064)^{\frac{1}{3}}$.

3. Compute:

(1) $a^{\frac{1}{4}}\bullet a^{\frac{1}{8}}\bullet a^{\frac{5}{8}}$; (2) $a^{\frac{1}{3}}\bullet a^{\frac{5}{6}}\div a^{-\frac{1}{2}}$; (3) $(x^{\frac{1}{2}}y^{-\frac{1}{3}})^6$;

(4) $4a^{\frac{2}{3}}b^{-\frac{1}{3}}\div\left(-\dfrac{2}{3}a^{-\frac{1}{3}}b^{-\frac{1}{3}}\right)$; (5) $\left(\dfrac{8a^{-3}}{27b^6}\right)^{-\frac{1}{3}}$.

4. (*Mental*) Is the following equation correct？If not, how can it be corrected?

(1) $a^{\frac{2}{3}}\bullet a^{\frac{3}{2}} = a$; (2) $x^{\frac{2}{3}}\bullet x^{-\frac{2}{3}} = 0$;

(3) $a^{\frac{2}{3}}\div a^{\frac{1}{3}} = a^2$; (4) $(a^{-\frac{1}{2}})^2 = a$.

3. Property of Surds

When *m*, *n* are positive integers, the operation rules of powers with fractional indicies can be formulated as follows

(1) $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}$ $\quad (a\ge 0,\ b\ge 0)$

(2) $\left(\dfrac{a}{b}\right)^{\frac{1}{n}} = \dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$ $\quad (a\ge 0,\ b>0)$

(3) $(a^{\frac{1}{n}})^m = a^{\frac{m}{n}}$ $\quad (a\ge 0)$

(4) $(a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}}$ $\quad (a\ge 0)$

As powers with fractional indices are related to surds, we can re-write the equations in surd form as follows

(1′) $\boldsymbol{\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}}$ $\quad (\boldsymbol{a\ge 0,\ b\ge 0})$

(2′) $\boldsymbol{\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}}$ $\quad (\boldsymbol{a\ge 0,\ b>0})$

(3′) $\boldsymbol{(\sqrt[n]{a})^m = \sqrt[n]{a^m}}$ $\quad (\boldsymbol{a\ge 0})$

(4′) $\boldsymbol{\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}}$ $\quad (\boldsymbol{a\ge 0})$

From the above equations, we can deduce and formulate operation rules of surds.

Equation (1′) means: Root of product of factors equals the product of root of its factors. For example,

$$\sqrt[3]{27\times 64} = \sqrt[3]{27}\times\sqrt[3]{64} = 3\times 4 = 12\,.$$

Equation (2′) means: Root of quotient equals root of its numerator divided by root of its denominator. For example,

$$\sqrt[3]{\frac{27}{64}} = \frac{\sqrt[3]{27}}{\sqrt[3]{64}} = \frac{3}{4}\,.$$

Equation (3′) means: Power of surd equals to the power of the base number under the root sign of the surd. For example,

$$(\sqrt[3]{5})^2 = \sqrt[3]{5^2} = \sqrt[3]{25}\,.$$

Equation $(4')$ means: Root of surd equals to the product of the root and the order of the surd, with the base number under the surd unchanged. For example,

$$\sqrt{\sqrt[3]{2}}=\sqrt[6]{2}\,,\quad \sqrt[3]{\sqrt[3]{2}}=\sqrt[9]{2}\,.$$

Looking at equation $(1')$ and equation $(2')$ in reverse, we observe that when multiplying (or dividing) two roots of the same order, we just take the product (or quotient) of the base numbers while keeping the order of the root unchanged. For example,

$$5\sqrt[3]{4}\bullet 2\sqrt[3]{2}=10\sqrt[3]{8}=20\,,\quad 5\sqrt[3]{4}\div 2\sqrt[3]{2}=\frac{5}{2}\sqrt[3]{2}\,.$$

When multiplying (or dividing) two surds with different orders together, we can transform the surds to the same order, and then take the product (or quotient) of the base unmbers accordingly. For example,

$$\sqrt{3}\bullet\sqrt[3]{2}=\sqrt[6]{27}\bullet\sqrt[6]{4}=\sqrt[6]{108}$$

$$\sqrt{3}\div\sqrt[3]{3}=\sqrt[6]{27}\div\sqrt[6]{9}=\sqrt[6]{3}$$

Using the above rules, we can operate mulitiplication, division, power or taking root of surds without problem.

According to equation $(1')$, we can move factors from inside the surd to the outside, or move factors from outside the root to the inside. For example,

$$\sqrt{a^2b}=\sqrt{a^2}\bullet\sqrt{b}=a\sqrt{b}$$

$$\sqrt[3]{a^6b^5}=\sqrt[3]{a^6\bullet b^3\bullet b^2}=\sqrt[3]{a^6}\bullet\sqrt[3]{b^3}\bullet\sqrt[3]{b^2}=a^2b\sqrt[3]{b^2}$$

$$x\sqrt[3]{y^2}=\sqrt[3]{x^3}\bullet\sqrt[3]{y^2}=\sqrt[3]{x^3y^2}$$

$$x^3\sqrt{y}=\sqrt{x^6}\bullet\sqrt{y}=\sqrt{x^6y}\qquad (x>0)$$

According to equation $(2')$, we can rationalize the surd in the denominator. For example,

$$\sqrt[3]{\frac{2}{27}}=\sqrt[3]{\frac{2}{3^3}}=\frac{1}{3}\sqrt[3]{2}$$

$$\sqrt[3]{\frac{3}{4}}=\sqrt[3]{\frac{3\times 2}{2^2\times 2}}=\frac{1}{2}\sqrt[3]{6}$$

First, the index of each factor inside the surd is less than the order of the surd;

Second, the base inside the surd does not contain denominator;

Third, the index of the base inside the surd is relatively prime to the order of the root.

An expression which fulfills the above three rules is called **the simplest surd**. For example, $a\sqrt[3]{a^2b}$ is a simplest surd, $\sqrt[3]{a^4b}$, $a\sqrt[4]{a^2b^2}$ and $a\sqrt[5]{\dfrac{b}{a^3}}$ are not simplest surds. When expressing an answer involving a surd, care should be taken to express the surd in the simplest surd form.

When surds are expressed in their simplest form, if the bases are the same, the root orders are the same, they are called **like surds**. For example,

$$\sqrt{12}=\sqrt{2^2\times 3}=2\sqrt{3}\,,\quad \sqrt[6]{27}=\sqrt[6]{3^3}=\sqrt{3}\,,\quad \sqrt{\frac{1}{3}}=\sqrt{\frac{1\times 3}{3\times 3}}=\frac{1}{3}\sqrt{3}\,,$$

Therefore $\sqrt{12}$, $\sqrt[6]{27}$, $\sqrt{\dfrac{1}{3}}$ are like surds. $\sqrt[3]{x}$, $\sqrt{x}$ are not like surds, and $4\sqrt[3]{a^2}$, $4\sqrt[3]{a}$ are also not like surds.

Addition and subtraction of surds is to group and combine various like surds. For example,

$$a\sqrt[n]{x}+b\sqrt[m]{y}-c\sqrt[n]{x}+d\sqrt[m]{y}=(a-c)\sqrt[n]{x}+(b+d)\sqrt[m]{y}\,.$$

Practice

1. Compute:

(1) $\sqrt{a^2b^4}$;　　(2) $\sqrt{121\times 64\times 256}$;

(3) $\sqrt[3]{a^9b^3t^{12}}$;　　(4) $\sqrt[3]{-343\times 512\times 729}$;

(5) $\sqrt[4]{16a^8b^{12}}$;　　(6) $\sqrt[n]{a^{2n}b^nc^{3n}}$.

Practice

2. Reduce the following fraction:

(1) $\sqrt{\dfrac{2}{81}}$; (2) $\sqrt{\dfrac{n}{49m^4}}$; (3) $\sqrt[3]{\dfrac{2}{27}}$;

(4) $\sqrt[4]{\dfrac{a^5}{16b^4}}$; (5) $\sqrt[3]{\dfrac{8x^3y^6}{27a^6b^9}}$; (6) $\sqrt[n]{\dfrac{a^nb^{2n}}{c^{3n}d^n}}$.

3. Compute:

(1) $(\sqrt[3]{a^2b})^2$; (2) $(3\sqrt[5]{a^4b^3})^2$;

(3) $(m\sqrt[4]{mn^2})^3$; (4) $\left(-\dfrac{x}{y}\sqrt{\dfrac{y}{x}}\right)$.

4. Compute:

(1) $\sqrt{\sqrt[3]{a^4b^2}}$; (2) $\sqrt[3]{2\sqrt{7}}$; (3) $\sqrt{a\sqrt[3]{a}}$; (4) $\sqrt[n]{2\sqrt{2}}$.

5. Reduce the number under the root sign by extracting as many factors as possible outside of the root sign:

(1) $\sqrt{8a^3}$; (2) $\sqrt{16t^5}$; (3) $\dfrac{1}{2}\sqrt{64p^3q^7}$;

(4) $\sqrt[3]{2t^4}$; (5) $\sqrt[3]{27a^5}$; (6) $\dfrac{1}{3}\sqrt[3]{27a^4b^5}$.

(7) $\sqrt[n]{a^{2n}b^{n+2}}$; (8) $\sqrt[4]{x^5-x^4y}$ $(x>y)$.

6. Rationalize the denominator of the following surd:

(1) $\sqrt{\dfrac{n^2}{8m}}$; (2) $\sqrt[3]{\dfrac{b^2}{9a^2}}$; (3) $\sqrt[3]{\dfrac{ax^3}{27m^2n^3}}$ (4) $\dfrac{1}{x}\sqrt[n]{\dfrac{1}{a^{n-2}}}$.

7. Simplify the following surd:

(1) $\sqrt{\dfrac{16c^3}{9a^5b}}$; (2) $\sqrt[3]{54a^4b^7}$;

(3) $x^2\sqrt[3]{\dfrac{3y}{2x^2}}$; (4) $n\sqrt[4]{\dfrac{1}{n^4}+\dfrac{1}{n^2}}$.

Practice

8. Compute:

(1) $\sqrt{8}+\sqrt[3]{54}-6\sqrt[3]{\dfrac{2}{27}}+3\sqrt{18}$;

(2) $7b\sqrt[3]{a}+5\sqrt{a^2x}-b^2\sqrt[3]{\dfrac{27a}{b^3}}-6\sqrt{\dfrac{b^2x}{9}}$.

【Example 7】 Compute the following using power with fractional indices:

(1) $\dfrac{a^2\bullet\sqrt[5]{a^3}}{\sqrt{a}\bullet\sqrt[10]{a^7}}$; (2) $(\sqrt[3]{5}-\sqrt{125})\div\sqrt[4]{5}$;

(3) $\sqrt[3]{xy^2(\sqrt{xy})^3}$.

Solution (1)
$$\frac{a^2\bullet\sqrt[5]{a^3}}{\sqrt{a}\bullet\sqrt[10]{a^7}}=\frac{a^2\bullet a^{\frac{3}{5}}}{a^{\frac{1}{2}}\bullet a^{\frac{7}{10}}}=a^{2+\frac{3}{5}-\frac{1}{2}-\frac{7}{10}}=a^{\frac{7}{5}}=\sqrt[5]{a^7}=a\sqrt[5]{a^2}$$

(2)
$$(\sqrt[3]{5}-\sqrt{125})\div\sqrt[4]{5}=(5^{\frac{1}{3}}-5^{\frac{3}{2}})\div5^{\frac{1}{4}}=5^{\frac{1}{3}-\frac{1}{4}}-5^{\frac{3}{2}-\frac{1}{4}}=5^{\frac{1}{12}}-5^{\frac{5}{4}}=\sqrt[12]{5}-\sqrt[4]{5^5}=\sqrt[12]{5}-5\sqrt[4]{5}$$

(3) $\sqrt[3]{xy^2(\sqrt{xy})^3}=\sqrt[3]{xy^2(x^{\frac{1}{2}}y^{\frac{1}{2}})^3}$

$=(x^{\frac{5}{2}}y^{\frac{7}{2}})^{\frac{1}{3}}$

$=x^{\frac{5}{6}}y^{\frac{7}{6}}$

$=\sqrt[6]{x^5}\bullet\sqrt[6]{x^7}$

$=y\sqrt[6]{x^5y}$

Except for special situations, it is usually simpler to perform multiplication, division, power and root taking operations using power with fractional indicies.

Practice

Compute:

(1) $2\bullet\sqrt{2}\bullet\sqrt[4]{2}\bullet\sqrt[8]{2}$; (2) $\dfrac{\sqrt{3}\bullet\sqrt[3]{9}}{\sqrt[6]{3}}$;

(3) $\sqrt{\dfrac{3y}{x}}\bullet\sqrt{\dfrac{3x^2}{y}}$; (4) $\dfrac{\sqrt{x}\bullet\sqrt[3]{x^2}}{x\bullet\sqrt[6]{x}}$;

(5) $\sqrt{\sqrt[3]{4}}$; (6) $\sqrt[3]{a\sqrt[4]{a^3}}$;

(7) $\dfrac{-3a^{\frac{2}{3}}b^{\frac{3}{4}}c^2}{9a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{3}{2}}}$; (8) $(x^{\frac{1}{3}}y^{\frac{3}{4}}-x^{\frac{1}{2}})x^{\frac{1}{2}}y^{\frac{1}{4}}$.

Exercise 10

1. Compute:

(1) $\left(\dfrac{1}{2}\right)^2+\left(\dfrac{1}{2}\right)^0+\left(-\dfrac{1}{2}\right)^{-2}$; (2) $(-3)^3+(-3)^{-3}+\left(-\dfrac{1}{3}\right)^{-3}-\left(-\dfrac{1}{3}\right)^3$

(3) $\left(\dfrac{1}{2}\right)^{-3}\times\left(-\dfrac{1}{3}\right)^2\times\left(\dfrac{1}{3}\right)^{-2}$; (4) $\left(\dfrac{b}{2a^2}\right)^3\div\left(\dfrac{2b^2}{3a}\right)^0\times\left(-\dfrac{b}{a}\right)^{-3}$.

2. Rewrite the following expression to remove powers with non-positive indices:

(1) $a^2b^{-1}c^3$; (2) $\dfrac{st^{-2}r}{u^{-1}v}$; (3) $\dfrac{2^{-2}m^{-2}n^{-3}}{3^{-1}m^{-3}n^3x^{-2}}$; (4) $\left(\dfrac{x+y}{2x-y}\right)^{-2}$

3. Rewrite the following expression to remove all denominators:

(1) $\dfrac{u+v}{u^4v}$; (2) $\dfrac{2x-y}{(x-y)(x+y)^2}$.

4. Compute:

(1) $(9a^2b^{-2}c^{-4})^{-1}$; (2) $5a^{-2}b^{-3}\div5^{-1}a^2b^{-1}\times5^{-2}ab^4c$;

(3) $\dfrac{a^{-3}+b^{-3}}{a^{-1}+b^{-1}}$; (4) $\dfrac{a^{-2}-b^{-2}}{a^{-2}+b^{-2}}$;

(5) $(a^{-1}+b^{-1})(a+b)^{-1}$; (6) $(x+x^{-1})(x-x^{-1})$.

5. Rewrite the following using scientific notation:
32000, 3200000, 3200000000, 0.000032, 0.0000032, 0.000000032, 483, 48.3, 4.83, 0.483, 0.0483, 0.00483.

6. "Si" is a unit of length used in manufacturing. 1 Si = 0.001 cm. The diameter of man's hair is about 7 Si. What is the equivalent length in cm? What is the equivalent length in m? Express answer in scientific notation.

7. The surface area of the Earth is approximately 149000000 km^2. Express it in scientific notation.

8. The radius of a germ is 4×10^{-5} m. Express it in decimal format (Unit is m).

9. One atom of Oxygen weighs 2.657×10^{-23} g, and one atom of Hydrogen weighs 1.67×10^{-24} g. Find the relative weight of one atom of Oxygen to that of Hydrogen (correct to 2 significant figures)?

10. Find the value of x in the following equation:

(1) $8=2^x$; (2) $\dfrac{1}{8}=2^x$; (3) $1=10^x$; (4) $0.1=10^x$;

(5) $3.4 = 3.4\times 10^{x}$; (6) $3400 = 3.4\times 10^{x}$;

(7) $0.034 = 3.4\times 10^{x}$; (8) $1 = 0.1^{x}$;

(9) $\dfrac{1}{64} = 2^{x}$; (10) $10 = 0.1^{x}$.

11. Write the following expression in surd form:

(1) $4^{\frac{1}{3}}$; (2) $y^{-\frac{2}{3}}$; (3) $a^{\frac{1}{2}}b^{-\frac{1}{2}}$; (4) $\dfrac{x^{-\frac{1}{4}}}{y^{-\frac{1}{4}}}$.

12. Compute:

(1) $(-2x^{\frac{1}{4}}y^{-\frac{1}{3}})(3x^{-\frac{1}{2}}y^{\frac{2}{3}})(-4x^{\frac{1}{4}}y^{\frac{2}{3}})$;

(2) $4^{\frac{1}{4}}(-3x^{\frac{1}{4}}y^{-\frac{1}{3}})\div(-6x^{-\frac{1}{2}}y^{-\frac{2}{3}})$;

(3) $\dfrac{-15a^{\frac{1}{2}}b^{\frac{1}{3}}c^{-\frac{3}{4}}}{25a^{-\frac{1}{2}}b^{-\frac{2}{3}}c^{\frac{5}{4}}}$; (4) $\left(\dfrac{16s^{2}t^{-6}}{25r^{4}}\right)^{-\frac{3}{2}}$.

13. Compute:

(1) $2x^{-\frac{1}{3}}\left(\dfrac{1}{2}x^{\frac{1}{3}}-2x^{-\frac{3}{2}}\right)$; (2) $(2x^{\frac{1}{2}}+3y^{-\frac{1}{4}})(2x^{\frac{1}{2}}-3y^{-\frac{1}{4}})$.

14. Compute:

(1) $\sqrt[4]{49x^{2}y^{2}}$; (2) $\sqrt[3]{\left(\dfrac{27p^{-6}}{p^{2}q^{-4}}\right)^{-2}}$;

(3) $\sqrt[5]{\dfrac{x}{y}\sqrt[4]{\dfrac{y}{x}}}$; (4) $\sqrt{x^{-3}y^{2}\sqrt[3]{xy^{2}}}$;

(5) $(\sqrt{a}\bullet\sqrt[3]{b^{2}})^{-3}\div\sqrt{b^{-4}a^{-1}}$; (6) $(\sqrt{3}-\sqrt[4]{243})\div 2\sqrt[3]{3}$.

15. Solve the following equation:

(1) $x-4\sqrt{x}+3=0$; (2) $\sqrt[3]{x}+\sqrt[3]{x^{2}}=2$;

(3) $\sqrt{x}-3\sqrt[4]{x}+2=0$.

Chapter Summary

I. The theme of this chapter is the concept and characteristics of power with zero index, negative integral index and fractional index

II. Definintion of power with zero index, negative integral index and fractional index are as follows:

$a^{0}=1 \quad (a\neq 0)$

$a^{-m}=\dfrac{1}{a^{m}} \quad (a\neq 0,\ m \text{ is positive integer})$

$a^{\frac{m}{n}}=\sqrt[n]{a^{m}} \quad (a\geq 0,\ m, n \text{ are both positive integers, } n>1)$

$a^{-\frac{m}{n}}=\dfrac{1}{a^{\frac{m}{n}}}=\dfrac{1}{\sqrt[n]{a^{m}}} \quad (a>0,\ m, n \text{ are both positive integers, } n>1)$

In this manner, we have extended the concept of power with positive integral index to power with rational index. All operation rules which apply to power with positive integral indices are also applicable to power with rational indices. Further

$$\frac{a^{m}}{a^{n}}=a^{m}\bullet a^{-n}=a^{m+(-n)} \qquad (a>0)$$

$$\left(\frac{a}{b}\right)^{n}=(a\bullet b^{-1})^{n}=a^{n}\bullet b^{-n} \quad (a>0,\ b>0)$$

Therefore, regarding powers with rational indices, both the operations of $\dfrac{a^{m}}{a^{n}}=a^{m-n}$ and of $\left(\dfrac{a}{b}\right)^{n}=\dfrac{a^{n}}{b^{n}}$ can be represented by the operation rule of $a^{m}\bullet a^{n}=a^{m+n}$ and of $(a\bullet b)^{n}=a^{n}\bullet b^{n}$. While there are 5 operation rules for powers with positive integral indices, there are only 3 operation rules for powers with rational indices. They are as follows:

$a^{m}\bullet a^{n}=a^{m+n} \quad (a>0,\ m,\ n \text{ are both ratioanl numbers})$

$(a^{m})^{n}=a^{mn} \quad (a>0,\ m,\ n \text{ are both ratioanl numbers})$

$(ab)^{n}=a^{n}b^{n} \quad (a>0,\ b>0,\ n \text{ is a ratioanl number})$

Although it is beond the scope of this book, we can briefly mention without detailed elaboration that the definition of power with rational indices can be extended to power with irrational indices, and further to power with real number indicies. The operation rules for power with rational indicies are also applicable to power with real number indices.

III. Representing a number in Scientific notation is to transform the number into the product of two parts: (i) the first part is a number with absolute value lying between 1 and 10 (can be 1) (ii) the second part is a power of 10. That means writing it in the form $\pm a\times 10^n$, where n is an integer, a is larger or equal to 1 and less than 10.

IV. Power with fractional indices is related to surds. Surds possess the following characteristics:

$(\sqrt[n]{a})^n = a$;

When n is an odd number, $\sqrt[n]{a^n} = a$;

When n is an even number, $\sqrt[n]{a^n} = |a| = \begin{cases} a & (a \ge 0) \\ -a & (a < 0) \end{cases}$

$\sqrt[np]{a^{mp}} = \sqrt[n]{a^m}$ $\quad(a \ge 0)$ (The basic property of surds);

$\sqrt[n]{ab} = \sqrt[n]{a} \bullet \sqrt[n]{b}$ $\quad(a \ge 0,\ b \ge 0)$;

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $\quad(a \ge 0,\ b > 0)$;

$(\sqrt[n]{a})^m = \sqrt[n]{a^m}$ $\quad(a \ge 0)$;

$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ $\quad(a \ge 0)$.

Addition and subtraction of surds would follow the algebraic operation rule of grouping like and unlike surds. ◦ Multiplication, division, power and taking roots of surds would be operated by following the operation rules of powers with fractional indices.

Revision Exercise 12

1. Compute:

(1) $\sqrt{7^2}$; (2) $\sqrt{(-7)^2}$; (3) $\sqrt{(x-y)^2}$ $(x > y)$;

(4) $\left(\sqrt{\frac{2}{15}}\right)^2$; (5) $\sqrt{0.49^2}$; (6) $\sqrt{a^2-14a+49}$ $(x < 7)$

2. Compute:

(1) $\sqrt{529\times 289}$; (2) $\sqrt{68.89\times 0.0009}$; (3) $\sqrt{65^2-16^2}$;

(4) $\sqrt{0.17^2-0.08^2}$; (5) $\sqrt{\frac{625}{1089}}$; (6) $\sqrt{\frac{0.49\times 121}{361\times 0.81}}$;

(7) $\sqrt{\frac{1.21}{4.41}\times 49}$; (8) $\sqrt{\frac{2.25x^6}{0.25y^2}}$ $(x > 0,\ y < 0)$.

3. Compute:

(1) $3\sqrt{a}+5\sqrt{b^3}+6\sqrt{a^5}-2\sqrt{b}$;

(2) $\sqrt{27x}+\sqrt{\frac{x}{3}}-\sqrt{0.03x}$;

(3) $a\sqrt{\frac{b}{a}}+b\sqrt{ab}-b\sqrt{\frac{1}{ab}}$;

(4) $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}-\sqrt{xy}+y\sqrt{\frac{1}{xy}}-x\sqrt{\frac{1}{xy}}$.

4. Compute:

(1) $\left(\sqrt{3ab}+\sqrt{\frac{b}{3a}}-\sqrt{\frac{27a}{b}}+\sqrt{\frac{a}{3b}}\right)\bullet\sqrt{3ab}$;

(2) $(3\sqrt{x}-\sqrt{5})(\sqrt{5}+3\sqrt{x})$;

(3) $(m-\sqrt{n})(\sqrt{n}+m)$;

(4) $(\sqrt{x}+\sqrt{y})(x+y-\sqrt{xy})$;

(5) $(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})$

5. Rationalize the denominator of the following fraction:

(1) $\dfrac{\sqrt{3}}{\sqrt{8}}$; (2) $\dfrac{a}{\sqrt{27a}}$; (3) $\dfrac{a-b}{\sqrt{a}+\sqrt{b}}$;

(4) $\dfrac{ab}{a\sqrt{b}+b\sqrt{a}}$; (5) $\dfrac{\sqrt{b}-a}{a+\sqrt{b}}$; (6) $\dfrac{1-xy}{\sqrt{x}+x\sqrt{y}}$;

(7) $\dfrac{x-1}{\sqrt{xy}+\sqrt{y}}$; (8) $\dfrac{1-xy}{\sqrt{\frac{1}{x}}+\sqrt{\frac{1}{y}}}$.

6. Find the value of x in the following equation:

(1) $35.23 = 3.523\times10^{x}$; (2) $3.523\times10^{x} = 0.003523$;

(3) $3.523\times10^{x} = 35230000$; (4) $3.523\times10^{x} = 3.523$.

7. The speed of light is 3×10^{5} km per second. The distance between the Sun and the Earth is 1.5×10^{8} km. Find the time for light to travel from the Sun to the Earh. (correct to one significant digit).

8. There are 6.02×10^{23} molecules in 18.00 g of water. Find the weight of one molecule of water, with answer expressed in scientific notation (correct to 2 significant figures).

9. Find the conditions for which the following expression has a meaning? :

(1) $\sqrt{x}$; (2) $\sqrt[4]{x-1}$; (3) $\dfrac{1}{\sqrt{x}}$; (4) $\sqrt[3]{-x}$

10. Compute:

(1) $(\sqrt[3]{-3.8})^{3}$; (2) $\sqrt[3]{-27}$;

(3) $\sqrt[5]{\dfrac{32}{243}}$; (4) $\sqrt[6]{(-5)^{6}}$;

(5) $\sqrt[4]{(1-a)^{4}}$ ($a>1$); (6) $\sqrt[8]{(m-n)^{8}}$ ($m<n$).

11. Find the value of $a+\sqrt[6]{(a-1)^{6}}$ according to the following condition:

(1) $a\ge 1$; (2) $a<1$.

12. Simplify the following expression to an expression with a single root sign:

(1) $\sqrt{x^{-3}y^{2}(\sqrt[3]{xy^{2}})}$ ($y\ge 0$);

(2) $\sqrt[3]{-2\sqrt{2}}$.

13. Simplify the following expression:

(1) $\dfrac{a^{\frac{1}{2}}-b^{\frac{1}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}+\dfrac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}$; (2) $\dfrac{(a+b)^{-1}-(a-b)^{-1}}{(a+b)^{-1}+(a-b)^{-1}}$;

(3) $\dfrac{a^{-2}-b^{-2}}{a^{-1}+b^{-1}}+\dfrac{1}{b}-\dfrac{1}{a}$; (4) $\left(\dfrac{e^{s}+e^{-s}}{2}\right)^{2}-\left(\dfrac{e^{s}-e^{-s}}{2}\right)^{2}$

(5) $(a^{2}-2+a^{-2})\div(a^{2}-a^{-2})$.

14. Compute:

(1) $125^{\frac{2}{3}}+\left(\dfrac{1}{2}\right)^{-2}+343^{\frac{1}{3}}-\left(\dfrac{1}{27}\right)^{-\frac{1}{3}}$;

(2) $\left(\dfrac{4}{9}\right)^{\frac{1}{2}}+(-5.6)^{0}-\left(2\dfrac{10}{27}\right)^{-\frac{2}{3}}+0.125^{-\frac{1}{3}}$;

(3) $\dfrac{(a^{-2}b^{-3})(-4a^{-1}b)}{12a^{-4}b^{-2}c}$;

(4) $(a^{2}b)^{\frac{1}{2}}\times(ab^{2})^{-2}\div\left(\dfrac{b}{a^{2}}\right)^{-3}$.

15. Find the value of x for the following equation:

(1) $5^{x}=125$; (2) $4^{x}=1$;

(3) $7^{x}=\sqrt[3]{7}$; (4) $\dfrac{1}{8}=2^{x}$;

(5) $\sqrt{3}=3^{x}$; (6) $\dfrac{1}{\sqrt[3]{10}}=10^{x}$.

(This chapter is translated to English and reviewed by courtesy of Mr. SIN Wing Sang, Edward.)