Chapter 14 Functions and their graphs

I. Rectangular Coordinate System

14.1 Plane rectangular coordinate system

We know that,on a straight line, if we fix the origin, the positive direction and the calibration of unit length, then we have a number axis (or number line). If we mark any point on this number axis, we know its position by reading the real number from the calibration. This real number is called the coordinate of the point on the number axis. Correspondingly, if we mark a point on a plane, how can we describe the position of the point on the plane?

If we ask people to drill a hole on a rectangular plate at a position M in the diagram 14-1, we can specify that the hole is at a distance 30 mm from the left side and 20 mm from the bottom. This specifies the exact postion of the hole M. Thus, we have shown that, on a plane, the position of a point can be specified by two real numbers.

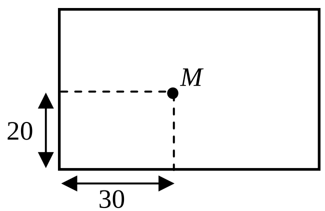


Diagram 14-1

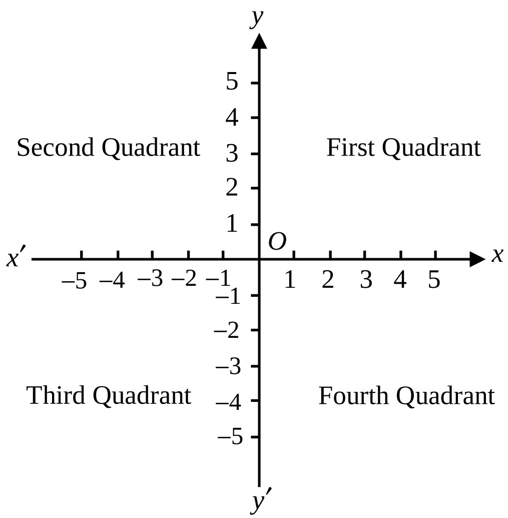


Diagram 14-2

On a plane, we draw two lines, xx' and yy', perpendicular to each other and crossing at the common point O (diagram 14-2). In general, xx' is the horizontal line, called the ***x*-axis**, with the direction to the right side of O taken to be the positive direction; while yy' is the vertical line called the ***y*-axis**, with the direction above O taken to be the positive direction. Usually both the x-axis and y-axis have the same calibration of unit length. Both the x-axis and y-axis are called coordinate axes. O is called the **origin**. In this manner, the origin, together with the two perpendicular coordinate axes, constitute a **plane rectangular coordinate system**. For simplicity we called it a **coordinate system**, and the plane on which we have established the coordinate system is called the **coordinate plane**.

The x-axis and the y-axis divide the coordinate plane into 4 sections, xOy、yOx'、$x'Oy'$、$y'Ox$，which are respectively called the first quadrant, the second quadrant, the third quadrant and the fourth quadrant. The boundaries of each quadrant are the x-axis and the y-axis, but the poins on the x-axis and the,y-axis do not belong to any of the quadrants.

After we have established the coordinate system on a plane, then any point on the plane can be uniquely associated with an ordered pair of real numbers. For example, for the point M (in diagram 14-3), draw a line through M perpendicular to the x-axis and note that the foot of the perpendicular line M_1 cuts the x-axis at the coordinate 3. Draw another line through M perpendicular to the y-axis and note that the foot of the perpendicular line M_2 cuts the y-axis at the coordinate 2. In this manner, there is a set of coordinates 3, 2

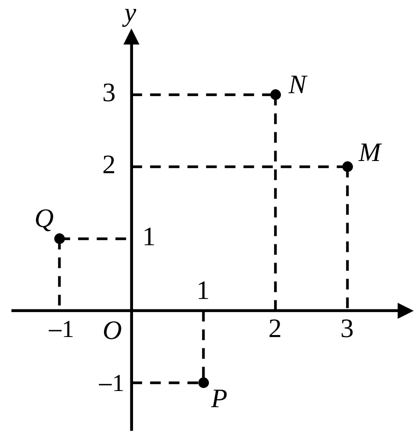


Diagram 14-3

corresponding to the point *M*. Here, 3 is the horizontal coordinate of *M*, and 2 is the vertical coordinate of *M*. Combined together, these two coordinates give the plane coordinates of the point M and are written succinctly as *M*(3, 2). It should be noted that the horizontal coordinate is always written before the vertical coordinate, and the two numbers are separated by a comma. That is, the plane coordinates of point *M* is an ordered pair of real numbers (called an ordered real number pair). In diagram 14-3, the plane coordinates of point *N* are (2, 3), written succincly as *N*(2, 3). From the diagram, we observe that M and N are two different points on the coordinate plane, and their coordinates (3, 2) and (2, 3) are the same two real numbers in different orders. Therefore the order is important in specifying the coordinates of points on the rectangular coordinate plane.

As an exercise, try to figure out the coordinates of point *P* and point *Q* in diagram 14-3.

In the converse, given any pair of ordered real numbers, there is a point on the coordinate plane corresponding to it. For example, for the ordered pair of real numbers (-3, -2), we can draw a line from the point -3 on the *x*-axis perpendicular to it, and draw another line from the point -2 on the *y*-axis perpendicular to it. The two perpendicular lines crossed at *A* and its coordinates are (-3, -2). Similarly, the ordered pairs of real numbers (3, 2), $(2.5, -1)$, $(-2, 0)$, (0, 0) correspond to points *B*, *C*, *D*, *O* (refer diagram 14-4).

From the above: we can conclude that any poin *M* on a coordinate plane corresponds to an ordered pair of real numbers ; on the other hand, any ordered pair of real numbers corresponds to a point *M* on the coordinate plane.

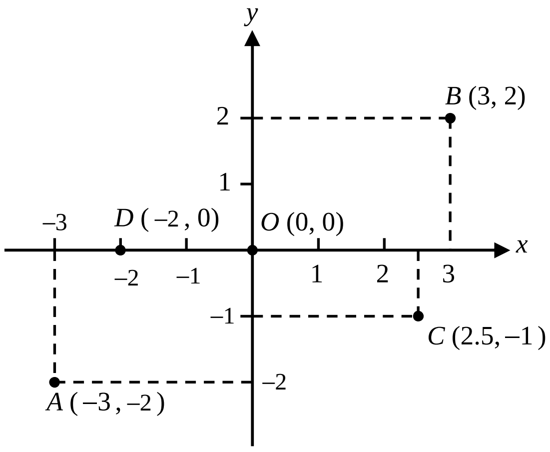
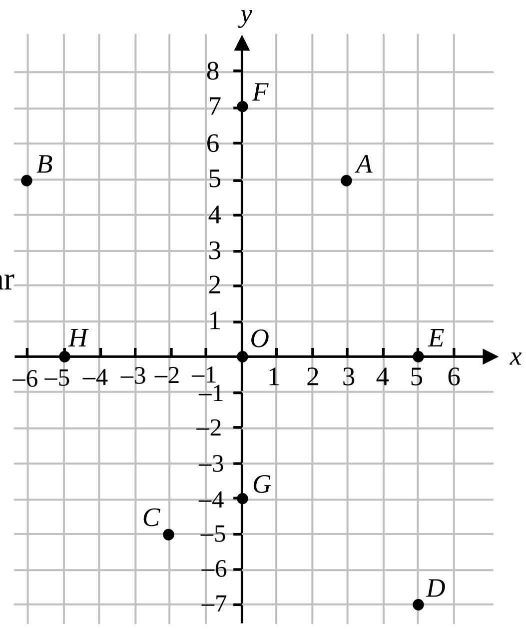


Diagram 14-4

Practice

1. Write down the coordinates of points *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *O* as read from the diagram.

2. Mark the following points on the rectangular corrdinate plane: $A(3, 6)$, $B(-1.5, 3.5)$, $C(-4, -1)$, $D(2, -3)$, $E(3, 0)$, $F(-2, 0)$, $G(0, 5)$, $H(0, -4)$.



(No. 1)

【**Example 1**】 In a rectangular coordinate system,

(1) what are the common characteristics of points on the *x*-axis?

(2) what are the common characteristics of points on the *y*-axis?

Solution (1) Look at diagram 14-5, take any point *P* on the *x*-axis, and drop a perpendicular line to the *y*-axis, the foot of the perpendicular line always falls on the orgin *O*(0,0). That means, the *y*-coordinate of *P* is 0. In other words, the *y*-coordinate of all points on the *x*-axis is 0.

(2) Similarly, the *x*-coordinate of all points on the *y*-axis is 0.

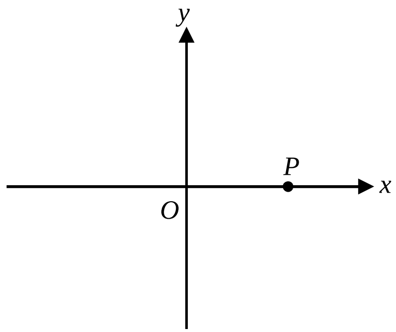


Diagram 14-5

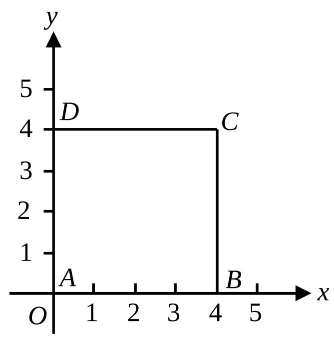


Diagram 14-6

【**Example 2**】In Diagram 14-6, given that the length of the sides of a square *ABCD* is 4, find the coordinates of the 4 vertices.

Solution The coordinates of the four vertices are,

$A(0, 0)$, $B(4, 0)$, $C(4, 4)$, $D(0, 4)$.

Practice

1. In example 2, write down the coordinates of the mid-points of the sides of the square.
2. Given that the length of the side of a square is 4, its diagonals intersect at the origin, and its sides are parallel to the rectangular coordinate axes, find the coordinates of the vertices.
3. Given that the coordinates of point *P* are (5, -3), what are the coordinates of the points, symmetric to *P* with respect to (i) the *x*-axis, (ii) the *y*-axis and (iii) the origin.
4. Draw a circle with the point (3, 0) as its centre, and with a radius of 5. Write down the coordinates of the points of intersection between the circle and the two axes.

14.2 Distance between two points

On a rectangular coordinate plane, we use coordinates to indicate the position of a point. Now we shall study how to use the coordinates of two points to calculate the distance between two points.

1. Distance between two points on the same number axis

Diagram 14-7 is a number axis, *A*, *B*, *C*, *D* are points on the axis. Their coordinates are 7, 2, -3, -7 respectively. Let us see whether we can determine the distance between two points on the number axis basing on the value of their coordinates.

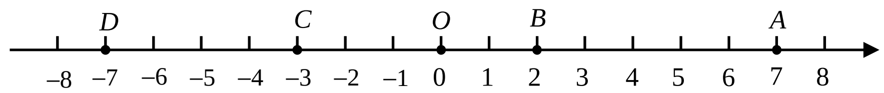


Diagram 14-7

From diagram 14-7 we can see, from line segments *BA*, *OA*, *OB*, that $BA = OA - OB$. Substituting the values, *OA* = 7, *OB* = 2, therefore $BA = 7 - 2$. But 7 and 2 are the coordinates of points *A* and *B* on the number axis. That means,the length of line segment *BA* is equal to the coordinate of point *A* minus the coordinate of point *B*. We know that, $7-2=|7-2|=|2-7|$. Using absolute value notation, we can say that the length of line segment *AB* (also means the distance between point *A* and *B*) is equal to the absolute value of the difference of the coordinates of *A* and *B*. Similarly

$$CB = OB + CO = 2 + 3 = |2-(-3)| = |(-3)-2|$$

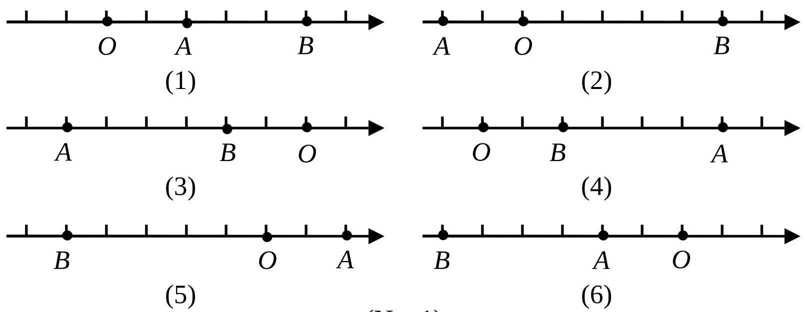
$$DC = DO - CO = 7 - 3 = |(-7)-(-3)| = |(-3)-(-7)|$$

In general, on a number axis, the distance between any two points, is equal to the absolute value of the difference of the coordinates of these two points. If the coordinates of the two points *A* and *B* are x_A and x_B respectively, then, the distance between ***A*** **and *B* can be determined by the following formula**

$$\boxed{\boldsymbol{AB = |x_B - x_A|}}$$

Practice

1. In diagrams (1) to (6) (each calibration on the number axis is of unit length)，fill in the table for the length of *AB*:



(No. 1)

Diagram	x_A	x_B	$x_B - x_A$	Length of *AB*
(1)				
(2)				
(3)				
(4)				
(5)	2	–5	–7	7
(6)				

2. On the number axis, the coordinates of points *A*, *B*, *C* and *D* are -5, 7, -2, 3 respectively. What is the distance (i) between *A* and *B*, (ii) between *B* and *C*, (iii) between *C* and *D*, (iv) between *C* and *A*?

2. Distance between any two points on a coordinate plane

Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be any two points on a coordinate plane (diagram14-8), from P_1 and P_2 draw lines P_1M_1 and P_2M_2 perpendicular to the *x*-axis, cutting the *x*-axis at $M_1(x_1, 0)$, $M_2(x_2, 0)$. Then from P_1 and P_2 draw lines P_1N_1 and P_2N_2 perpendicular to the *y*-axis, cutting the *y*-axis at $N_1(0, y_1)$, $N_2(0, y_2)$. The straight lines P_1N_1 and P_2M_2 intersect at point *Q*.

Because $\triangle P_1QP_2$ is a right-angled triangle, applying pythagoras' theorem, we get

$$P_1P_2^2 = P_1Q^2 + QP_2^2$$

$\because$ $P_1Q = M_1M_2 = |x_2 - x_1|$

$QP_2 = N_1N_2 = |y_2 - y_1|$

$\therefore$ $P_1P_2^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

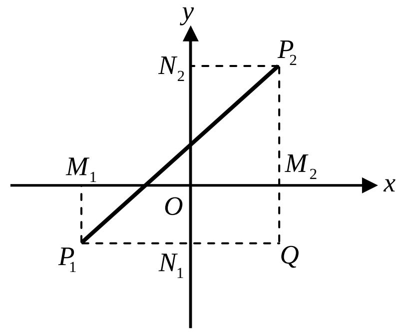


Diagram 14-8

Hence we obtain the following formula to compute the distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the rectangular coordinate plane:

$$\boxed{P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

【**Example 1**】 Find the distance between the two points $P_1(-3, 5)$, $P_2(1, 2)$.

Solution $x_1 = -3$, $y_1 = 5$; $x_2 = 1$, $y_2 = 2$.

Substituting the values into the distance formula, we get

$$P_1P_2 = \sqrt{[1-(-3)]^2 + (2-5)^2} = \sqrt{4^2 + (-3)^2} = 5.$$

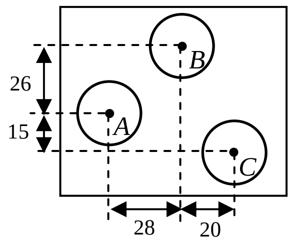


Diagram 14-9

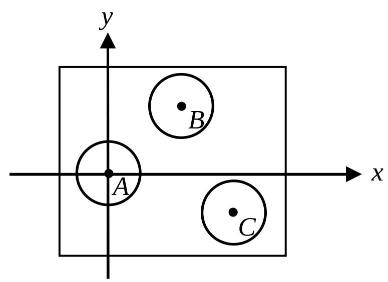


Diagram 14-10

【**Example 2**】Diagram 14-9 shows a specification of some spare parts (unless specified otherwise, all measurements used in this book are in mm). A rectangular coordinate system is established as in diagram 14-10. Find the distance between A and B and the distance between B and C (accurate to 0.01mm)

Solution The coordinates of the respectivee points are $A(0, 0)$, $B(28, 26)$, $C(48, -15)$.
Substituting the coordinates of the points into the distance formula, we get

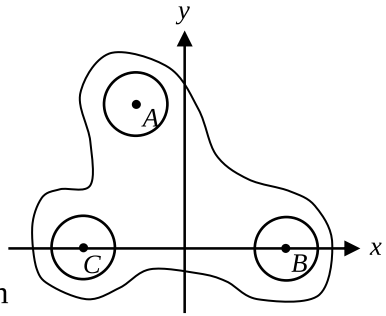
$$AB = \sqrt{(28-0)^2 + (26-0)^2} = \sqrt{1460} \approx 38.21$$

$$BC = \sqrt{(48-28)^2 + (-15-26)^2} = \sqrt{2081} \approx 45.62$$

That is, the distance between A and B is 38.21 mm, and the distancet between B and C *is* 45.62 mm。

Practice

1. Find the distance between the two points in each of the following:
   (1) $P_1(-1, 0)$、$P_2(2, 0)$； (2) $P_1(0, 6)$、$P_2(0, -2)$；
   (3) $A(-2, 0)$、$B(-4, 3)$； (4) $A(2, -5)$、$C(2, 3)$；
   (5) $M(-3, 8)$、$N(-1, -2)$； (6) $O(0, 0)$、$P(2, -3)$。
2. Refer to diagram. Given that the coordinates fo the points are $A(-20, 50)$, $B(40, 0)$, $C(-40, 0)$, find the distance between each two points (accurate to 0.01 mm).

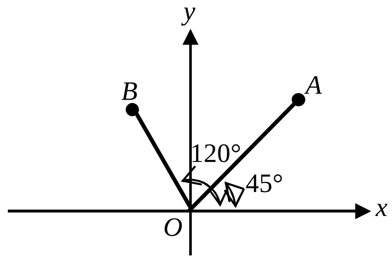


(No. 2)

3. Ship A is stationed at 50 km East and 30 km North of a port. Ship B is stationed at 17 km East and 26 km South of the same port. Establish a suitable plane coordinate system and use it to determine the distance between A and B.

Exercise 3

1. Plot the points whose x-coordinates are -4, -3, -2, -1, 0, 1, 2, 3, 4, while their y-coordinates are given by the function $y = x^2$. Connect these points by a smooth curve.
2. Refer to the diagram on the right. In the diagram $OA = 8$, $OB = 6$. Find the coordinates of points A, B.

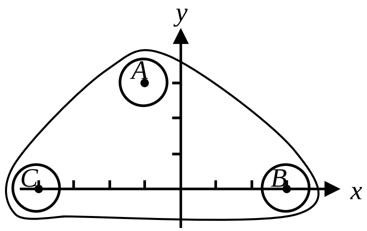


(No. 2)

3. (1) Given that point $P(x, y)$ lies in the first quadrant, what are the signs of the values of x, y?
   (2) Given that point $Q(x, y)$ lies in the third quadrant, what are the signs of the values of x, y?
4. In the first quadrant, there is a line bisecting the angle between the x-axis and y-axis. For any point on the line bisector, what is the relationship between its x-coordinate and its y-coordinate? How about the line bisector in the second quadrant?
5. The length of a side of a rhombus is 5, and the length of one of the diagonals is 6. If the two diagonals form the x-axis and the y-axis of the rectangular coordinate system, what are the coordinates of the 4 vertices (Note: there are two sets of answers)?
6. Given a point $P(a, b)$, what is the coordinates of points symmetric to P with respect to (i) the x-axis (ii) the y-axis and (iii) the origin.
7. The coordinates of points A and B on a number axis are x_A and x_B. Find the distance between A and B with regard to the following values of x_A and x_B.
   (1) $x_A = 8$, $x_B = 6$; (2) $x_A = 2$, $x_B = -1$;
   (3) $x_A = -3$, $x_B = 0$; (4) $x_A = 0$, $x_B = -8$.

8. Find the length of the sides of the triangle formed by the points A, B and below, and determine whether the triangle is an isosceles triangle, an equilateral triangle, or a right-angled triangles.
   (1) $A(-3, 0)$, $B(3, 0)$, $C(0, 3\sqrt{3})$;
   (2) $A(-4, 3)$, $B(2, -5)$, $C(0, 6)$;
   (3) $A(5, 1)$, $B(2, -2)$, $C(2.5, 0.5)$;
   (4) $A(3, 0)$, $B(6, 4)$, $C(-1, 3)$.

9. Refer to the diagram on the right. Calculate the distance between the centres of the three holes, given that their coordinates are $A(-10, 30)$, $B(30, 0)$, $C(-40, 0)$.



(No. 9)

II. Functions

14.3 Functions

1. Constants and variables

Let us study the following examples:

(1) A train runs at a speed of 60 km/h. The distance s(km) traveled by the train is related to the time t(h) taken by the function $s = 60t$

(2) The area A (cm^2) of a circle is related to its radius r (cm) by the function $A = \pi r^2$

In example (1), when we use the function $s = 60t$ to calculate the distance traveled by the train at different durations of time, we notice that both the values of t and s may change, but the value of speed is always the same. In example (2) when we use the function $A = \pi r^2$ to calculate the area of a circle for different lengths of the radius, we notice that both the values of r and A may change , but the value of π is always the same.

In the calculation, any element which can change to different values is called a variable, such as the elements of t hour, s km, r cm, A cm^2 in the above examples. On other hand, any element whose value remains changed is called a constant, such as the elements of 60 km/h and π. The classification of which element is a constant and which element is a variable depends on the circumstance and purpose of calculation. We can see that in example (1), when the spped is fixed, the speed is a constant, while the distance and the time are variables. On the other hand, when the time is fixed, time is a constant while the speed and distance are variables.

2. Functions

In example (1) above, the variable of time t must take a non-negative value (positive real number or zero). For each value of t, there is a unique value of s corresponding to it. Examples are as follows:

t (hr)	1	1.5	2	2.5	3	$\cdots$
s (km)	60	90	120	150	180	$\cdots$

Similarly, in example (2), the radius r must be a positive real number. For each value of r, there is a unique value of A corresponding to it.

This kind of relationship between variables is very common in practice, such as in agriculture production and in scientific experiment. Some specific examples are as follows:

(3) The water storage of a certain reservoir is related to the depth of water in the reservoir, with measurements recorded in the following table:

Depth of water h (m)	0	5	10	15	20	25	30	35
Water storage Q (10^4 m^3)	0	20	40	90	160	275	437.5	650

Having obtained this table, we can tell the water storage for any value of the depth of water *h*, selected from the first row of the table. When the value of the depth of water is selected, there is a unique value of water capacity *Q* corresponding to it. For example: if $h = 20$ (m), then $Q = 160\,(10^4 \mathrm{m}^3)$; $h = 30$ (m), $Q = 437.5\,(10^4 \mathrm{m}^3)$.

(4) Diagram 14-11 is a daily temperature graph plotted by an automatic temperature recorder of a weather station.

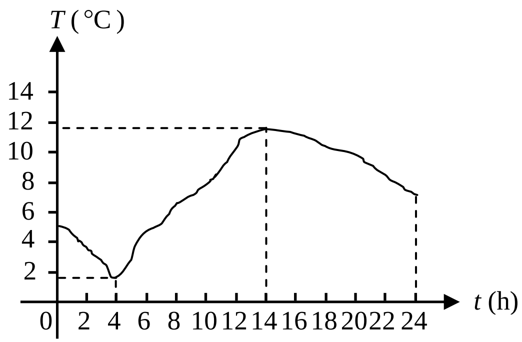


Diagram 14-11

The graph reflects the relationship between the air temperature variable T (°C) and the time variable t (h). Having obtained this graph, we can tell the air temperature at any selected time of the day. For any value of time t between 0 and 24, there is a unique value of air termperature T corresponding to it. For example: $t = 4$ (h), $T = 1.8$ (°C); $t = 14$ (h), $T = 11.8$ (°C).

In a changing environment involving two variables *x* and *y*, if for each value of *x* taken from a set of values,, there is a unique value of *y* corresponding to it, then *y* is a **function** of *x*, and *x* is called the **independent variable**. For example, distance *s* is a function of time *t*; area of a circle *A* is a function of radius *r;* water storage *Q* is a function of depth of water *h*, air temperature *T* is a function of time *t*.

We observe that, both $60t$ and πr^2 are algebraic expression with one alphabet as variable. In general, the value of this kind of single variable algebraic expression is depending on the value of the independent variable. It is possible to use different alphabet for the independent variable so long as the representation makes sense and does not cause confusion to the study. For each value of the independent variable, the expression has a unique value corresponding to it. Hence every expression involving only one alphabet is a function of the alphabet. For example, $x - 2$ is a function of *x*, $\dfrac{1}{1-u^2}$ is a function of *u*, $\dfrac{1}{\sqrt{t^2-5}}$ is a function of *t*, etc..

【Example 1】 Find the domain of possible values of the independent variable *x* in the following function:

(1) $y = 2x + 3$; (2) $y = -3x^2$;

(3) $y = \dfrac{1}{x-1}$; (4) $y = \sqrt{x-2}$.

Solution (1) For x to take the value of any real number, the function $2x+3$ has meaning. Therefore the domain of possible values of the independent variable *x* can be any number in the real number set.

(2) For x to take the value of any real number, the function $-3x^2$ has meaning. Therefore, the domain of possible values of the independent variable *x* can be any number in the real number set.

(3) When $x = 1$, the function $\dfrac{1}{x-1}$ is not meaningful; when $x \neq 1$, the function $\dfrac{1}{x-1}$ has meaning. Therefore the domain of possible values of independent variable *x* can be any number other than 1 in the real number set.

(4) When $x < 2$, the function $\sqrt{x-2}$ is not meaningful; when $x \geq 2$, the function $\sqrt{x-2}$ has meaning. Therefore the domain of possible values of the independent variable *x* can be any real number greater than or equal to 2.

Note: In the function $A = \pi r^2$, from an algebraic point of view, r can be any real number. But from a practical point of view, r being the length of the radius of a circle, can only take positive real number value (greater than 0). This implies that when considering the domain of possible values of the independent variable, we have to take into consideration the practical situation to ensure that the result is meaningful.

【**Example 2**】In example 1, when $x = 2$, what is the corresponding value of *y*.

Analysis: Since $x = 2$ is within the domain of possible values of *x*, we can substitute *x* = 2 in the function to calculate the value of *y*.

Solution (1) $y = 2\times 2 + 3 = 7$;

(2) $y = -3\times 2^2 = -12$;

(3) $y = \dfrac{1}{2-1} = 1$;

(4) $y = \sqrt{2-2} = 0$.

When the independent variable takes on a value within the domain of possible values, say $x = a$, then there is a unique value of the function corresponding to it, which we call the function value at $x = a$. Example 2 requires us to find the function value at $x = 2$.

Practice

1. (*Mental*) Identify which are variables and which are constants in the following equation?
   (1) Distance formula under constant verlocity $s = vt$, where *v* is the velocity, *t* is time and *s* is distance covered;
   (2) Volume of sphere formula $V = \dfrac{4}{3}\pi r^3$, where *r* is the radius and *V* is the volume of the sphere;
   (3) The internal angle of an n-side regular polygon is given by this formula:
   $$\alpha = \frac{(n-2)180°}{n}.$$

Practice

2. Find the doman of possible values of the independent variable *x* in the following funtion:
   (1) $y = \dfrac{x-1}{2}$; (2) $y = \dfrac{3}{x-4}$;
   (3) $y = -\sqrt{x-5}$; (4) $y = \dfrac{1}{x^2 - x - 2}$.
3. In Problem 2 above, find the value of each of the functions (i) when $x = 9$; (ii) when $x = 30$.

14.4 Different ways to represent a function

There are three common ways to represent a function.

1. Analytical method which uses an equation to show how a dependent variable can be determined from the value of an independent variable. This equation is called the analytical functional expression (or analytical relationship equation), abbreviated as analytical formula, For example $s = 60t$, $A = \pi r^2$, $y = \sqrt{x-2}$, $V = \dfrac{4}{3}\pi r^3$, $s = \dfrac{1}{1-u^2}$, $z = \dfrac{1}{\sqrt{t^2-5}}$ etc..

2. Tabular method which uses a table to show the value of an independent variable and the corresponding value of the dependent variable of a function. In section 14.3, example (3) is a tabular method of writing the function of water storage in terms of the depth of water in the reservoir. Similar examples are mathematical tables like the square table, the square root table, and the logarithmic table.

3. Graphical method which treats the independent variable *x* and the dependent variable *y* as an ordered pair and plots the corresponding point on the rectangular coordinate plane. By connecting all the points together to form a graph, the function is depicted Thus the graphical method represents a function by showing how the dependent variable is related to the independent variable on the rectangular coordinate plane. The weather chart in section 14.3 is an example.

When the analytical expression of a function is known, we can draw a graph of this function by applying the following three steps, namely (i) tabuling the values, (ii) plotting the points, (iii) connecting the points using one or more smooth curves (which include stragith lines). This method is called sketching the graph, which may be an approximation only. To get a closer approximation, we shall have to tabulate more point values and draw the graph more finely.

【Example 1】 Draw the graph of the function $y=\frac{1}{8}x^3$.

Solution 1. Tabulate the values. Pick certain values of x from its doman, calculate the corresponding values of the dependent variable, and tabulate the data in the form of a table, as follows:

x	⋯	–4	–3	–2	–1	0	1	2	3	4	⋯
$y=\frac{1}{8}x^3$	⋯	–8	–3.38	–1	–0.13	0	0.13	1	3.38	8	⋯

2. Plot the points on the rectangular coordinate plane.

3. Connect the points by a smooth curve to form a graph for the function $y=\frac{1}{8}x^3$ (Diagram 14-12).

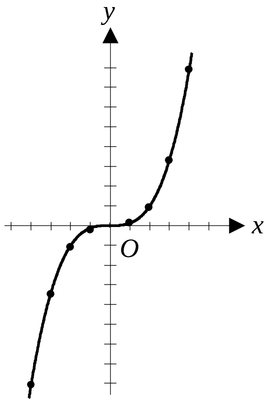
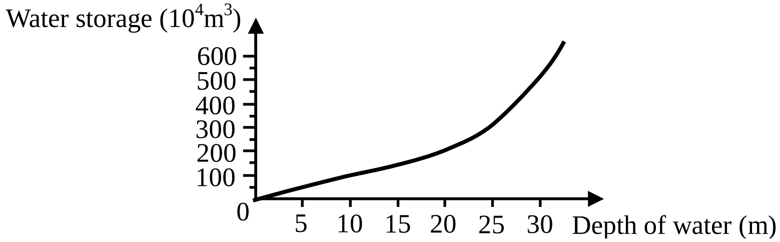


Diagram 14-12

Practice

1. Write an analytic expression for the following function:
   (1) It is known that for every 100 m rise in altitude h (km), the air temperature drops by $0.6\,°\mathrm{C}$. Write the function to express T ($°\mathrm{C}$) in terms of h (km);
   (2) A factory has a stock of 1500 T of coal. Write the function to express the number of days, y, the stock will last in terms of the daily average usage of x T of coal.
2. Fill in the tablefor the water storage corresponding to the deapth of water in the reservoir:

Depth of water (m)	5	10	15	20	25
Water storage (10^4 m^3)					



(No. 2)

3. Draw a graph for the following function:
   (1) $y=x$;   (2) $y=x+1$.

Exercise 4

1. Based on the following procedural steps of finding the value of y from x, develope a function to represent the relation of y in terms of x:

   (1) $\boxed{\text{Input } x} \longrightarrow \boxed{\times 3} \longrightarrow \boxed{-1} \longrightarrow \boxed{\text{Output } y}$

   (2) $\boxed{\text{Input } x} \longrightarrow \boxed{-2} \longrightarrow \boxed{\times 3} \longrightarrow \boxed{+1} \longrightarrow \boxed{\text{Output } y}$

2. Find the domain of possible values of x in the following function:

(1) $y = 3x^2 - 5x + \sqrt{3}$; (2) $y = \dfrac{2x+1}{x-2}$;

(3) $y = \dfrac{x+1}{x^2 - x - 6}$; (4) $y = \dfrac{3x}{4x^2 - 9}$;

(5) $y = \sqrt{2x-5}$; (6) $y = x + \sqrt{x+2}$;

(7) $y = \dfrac{x+2}{x^2 + 5x + 6}$.

3. For the function $y = x^2 - 3x + 4$, fill up the following table for the corresponding values of y:

x	-2	-1	0	1	$1\frac{1}{2}$	2	3	4
y								

4. For the function $y = \dfrac{2x+1}{x-2}$, find the value of the function when $x = 3$, -4, 0, $-\dfrac{1}{2}$, $\sqrt{2}$ respectively. When $x = a^2 + 3$, what is the value of y?

5. For the function $y = 2x^2 - 5x + 3$, what are the values of the function when $x = 0$ and 2 respectively. What value of x will cause the value of the function to become 0?

6. At $0°\text{C}$, the volume of a copper is 1000 cm^3. For each $1°\text{C}$ rise in temperature, the volume of the sphere will increase by 0.051 cm^3. Write the function of volume V in terms of temperature T, Use the function to find the volume of the sphere when temperature is increased to $200°\text{C}$.

7. For an isosceles triangle, write a function of its apex angle y in terms of the base angle x. Find the domain of possible values of x.

8. Given that x and y are related by the following equation. Write the function of y in terms of x:

(1) $2x + 4y = 12$; (2) $xy = 15$;

(3) $(x-2)(y+3) = -6$; (4) $x = \dfrac{3y+2}{4y-3}$;

(5) $y^2 = 4x$ ($y \geq 0$); (6) $y - \dfrac{2}{3}x = 0$.

9. For the function $y = ax + b$ (where a, b are constants). When $x = 1$, $y = 7$. When $x = 2$, $y = 16$. Find the values of a and b.

10. The length of a spring and its attached weight are tabulated in the following table:

x (kg)	0	1	2	3	4	5	6	7	8
y (cm)	12	12.5	13	13.5	14	14.5	15	15.5	16

Assuming that y is related to x by the function $y = ax + b$ (a and b are constants), Use any two pairs of values in the table to calculate the valus of a and b. Use other two pairs of data to see if the same result can be obtained.

11. The following table records the temperature at different hours on a certain day:

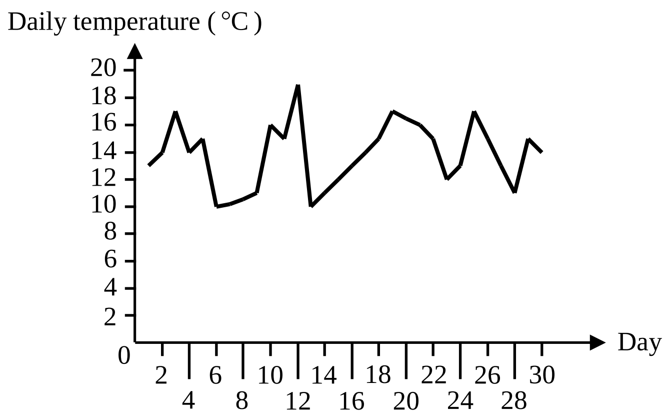
Time(hr)	0	2	4	6	8	10	12	14	16	18	20	22	24
Temperature (°C)	-2	-3	-4	0	4	7	9	10	8.5	7	3.5	1	-1

Plot a graph of time and temperature based on the values recorded in the table.

12. The following graph shows the average daily temperature of a certain month. Based on this graph, find the following:

(1) The highest and the lowest average daily temperature in the month;

(2) The biggest amplitude of change of average daily temperature in the month.



(No. 12)

13. Draw a graph for the following function:

(1) $y = 4x$; (2) $y = 3x + 1$;

(3) $y = -2x^2$; (4) $y = 2x^2 - 1$.

III. Directly Proportional Function and Inversely Proportional Function

14.5 Directly Proportional function and its graph

1. Directly proportional function

Let us study the following examples:

(1) To apply fertilizer to a field, each acres of field will require 1.5 kg of fertilizer. The function showing the relationship of weight of fertilizer *y* (kg) required in terms of the area of the field *x* (acres). is

$$y = 1.5x\ .$$

(2) The density of copper is 8.9 g/cm^3, The function between the weight of copper *W* (g) in terms of its volume *V* (cm^3) is

$$W = 8.9V\ .$$

In example (1), the relationship of variables *y* divided by *x*, namely $\frac{y}{x}$ is a constant. Similarly, in example (2), the relationship of *W* divided by *V* is also a constant. In mathematics, this relationship is called directly proportional. Here we call functions like $y = 1.5x$, $W = 8.9V$ directly proportional functions.

In general, function like $y = kx$ (*k* is a constant not equal to zero) is called directly **proportional function** (variables *y* and *x* are directly proportional). The constant *k* is called the scale factor of variables *y* and *x* . In arithmetic, *k* is restricted to a positive number. Now we can extend the concept for k to take any non-zero number, positive or negative. Once the value of the scale factor *k* is known, the directly proportional function is specified.

【Example 1】 The circumference *C* of a circle is directly proportional to its radius *r*. It is known that when $r = 2$ cm, $C = 12.56$ cm.

(1) Find the function C in terms of *r*;

(2) Find the circumference of the circle when radius is 3.5.

Solution (1) As *C* and *r* are directly proportional, we have

$$C = kr\ .$$

Substituting $r = 2$, $C = 12.56$ into the equation, we have

$$12.56 = 2k$$

$$k = 6.28$$

$$\therefore \quad C = 6.28r$$

(2) When $r = 3.5$,

$$C = 6.28 \times 3.5 = 21.98\ .$$

2. Directly proportional function and its graph

Let us plot the graph of the function $y = 2x$.

From the domain of possible values of x, pick some values of x, calculate the corresponding value of y for each value of x, and tabulate the result as follows:

x	$\cdots$	-2	-1	0	1	2	$\cdots$
y	$\cdots$	-4	-2	0	2	4	$\cdots$

For each pair of values in the table, we can plot the point on the rectangular coordinate system. Connect the points together to form a graph. We note that the graph of the function $y = 2x$ is a straight line passing through, inter alia, the points O(0, 0), and A(1, 2). (refer diagram 14-13).

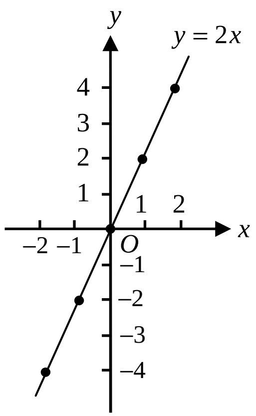


Diagram 14-13

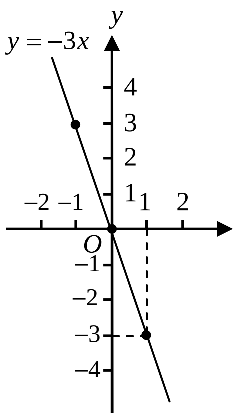


Diagram 14-14

Similarly, the graph of $y = -3x$ is also a straight line passing through the points O(0, 0) and $B(1, -3)$ (referdiagram 14-14).

In general, directly proportion function $y = kx$ is a straight line passing through the points O(0, 0) and A(1, k). From now on we can describe the directly proportional function $y = kx$ as the straight line $y = kx$.

Because the graph of a straight line can be determined by two points, when plotingt the straight line $y = kx$, we do not need to mark more than two points. Usually we shall use the two points O(0, 0) and A(1, k) for convenience.

【Example 2】 On the same rectangular coordinate plane, draw the graphs of the following functions:

$$y = 2x, \quad y = x, \quad y = \frac{1}{2}x.$$

Solution Because O(0, 0), A(1, 2) are on the straight line $y = 2x$, we can plot these two points and connect them by a straight line. This straight line is the graph of the function $y = 2x$. Similarly, the straight line passing through O(0, 0) and B(1, 1) is the graph of the function $y = x$.The straight line passing through O(0, 0), $C(1, \frac{1}{2})$ is the graph for the function $y = \frac{1}{2}x$ (refer diagram 14-15).

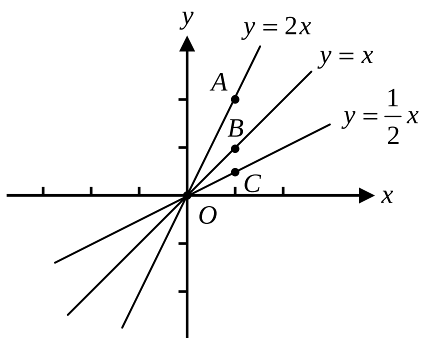


Diagram 14-15

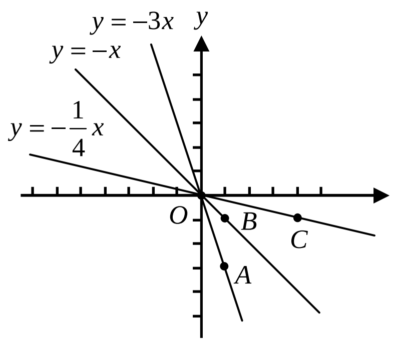


Diagram 14-16

【Example 3】 On the same rectangular coordinate plane, draw the graphs of the following functions:

$$y = -3x, \quad y = -x, \quad y = -\frac{1}{4}x.$$

Solution The straight line passing through the points O(0, 0) amd $A(1, -3)$ is the graph for the function $y = -3x$. The straight line passing through O(0, 0), $B(2, -2)$ is the graph for the function $y = -x$. The straight line passing through O(0, 0), $C(4, -1)$ is the graph for the function $y = -\frac{1}{4}x$

(refer diagram 14-16).

From diagram 14-14 and diagram 14-15 we obaserve that directly **proportional function $y = kx$ has the following characteristics :**

When $k > 0$, the graph $y = kx$ is a straight line lying inside the first and third quadrants. *y* increases as *x* increases.

When $k < 0$, the graph $y = kx$ is a straight line lying inside the second and the fourth quadrants. *y* decreases as *x* increases.

Practice

1. (*Mental*) Given that x is an independent variable, which of the following function is a directly proportional function? Which one is not a directly proportional function? Why?

   (1) $y = -8x$; (2) $y = \frac{-8}{x}$; (3) $y = 8x^2$; (4) $y = 8x + 1$.
2. Is the function of the area of circle in terms of its radius a directly proportional function?
3. Given that variables y and x are directly proportional. When $x = 2$, $y = 15$. Find the scale factor between y and x and write down the function of *y* in terms of *x*.
4. In the same rectangular coordinate plane, draw the graphs of the following functions:

$$y = \frac{2}{3}x,\quad y = -\frac{2}{3}x,\quad y = \frac{3}{2}x,\quad y = -\frac{3}{2}x.$$

14.6 Inversely proportional function and its graph

1. Inversly proportional function

Let us study the following examples:

(1) The area of a rectangle is 12 cm^2. The function between the base, y (cm) and height, x (cm) is governed by the following equation

$$y = \frac{12}{x}.$$

(2) To run a distance of 25 km, the function between the time required, t (hour) and the average speed, v (km/ hour) is governed by the following equation

$$t = \frac{25}{v}.$$

In example (1), the product of the two variables y and x is a constant (equal to 12). Similarly, in example (2), the product of the two variables t and v is also a constant. In arithmatic, we describte the two variables as inversely proportional. Here we describe the functions $y = \frac{12}{x}$, $t = \frac{25}{v}$ as inversely proportional functions.

In general, the function $y = \frac{k}{x}$ (k is a constant not equal to zero) is called an **inversely proportional function** (we say that y and x are **inversely proportional**). In arithmatics, k can only be a poisitive number. Here we can extend the concept for k to take any non-zero numbers, positive or negative numbers. When k is known, the inversely proportional function is fully specified.

【**Example 1**】Given that the volume of a cylinder is unchanged, and that when the height $h = 12.5$ cm, its base area $S = 20$ cm^2.

(1) Find the function of S in terms of h;

(2) When $h = 5$ cm, find its base area S.

Solution (1) If the volume of the cylinder does not change. the base area and its height h are inversely proporational. Therefore

$$S = \frac{k}{h}.$$

Substituting $h = 12.5$, $S = 20$ into the equation, we get

$$20 = \frac{k}{12.5}$$
$$k = 250$$

Answer: the function is $S = \frac{250}{h}$

(2) When $h = 5\,\text{cm}$,

$$S = \frac{250}{h} = \frac{250}{5} = 50\,(\text{cm}^2)$$

Answer: when height is 5cm, the base area is 50cm^2

2. Graph of inversely proportional function

【**Example 2**】Draw of graph of inversely proportional functionss

$$y = \frac{6}{x} \text{ and } y = -\frac{6}{x}.$$

Solution The domain of possible values of x consists of all non-zero real numbers. Pick some non-zerp values for *x*. For each value of *x*, calculate the corresponding value of *y* . Tabulate the results in the following table :

x	⋯	–6	–5	–4	–3	–2	–1	1	2	3	4	5	6	⋯
y	⋯	–1	–1.2	–1.5	–2	–3	–6	6	3	2	1.5	1.2	1	⋯

Regard each pair data as the coordinates of points on the rectangular coordinate plane. Plot the points and connect the points in the first quadrant together. This gives one part of the graph. Then connect the points in the third quadrant. This gives another part of the graph. These tow parts together form the graph of the function $y = \frac{6}{x}$ (refer diagram14-17).

Applying the same method, we can draw the graph of the function $y = -\frac{6}{x}$ (refer diagram 14-18).

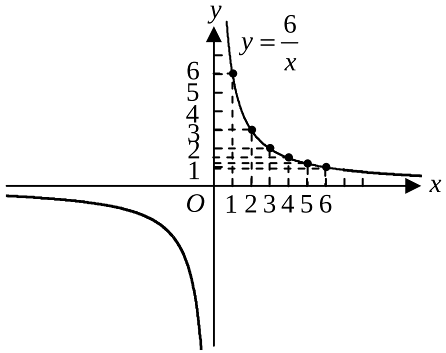


Diagram 14-17

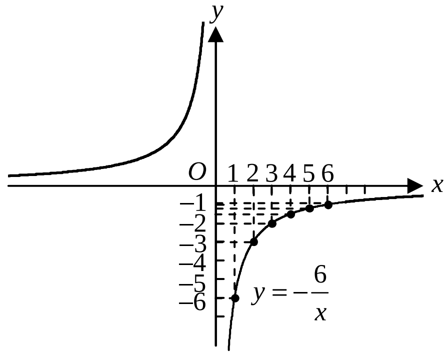


Diagram 14-18

In general, the graph of an inversely proportional function $y = \frac{k}{x}$ ($k \neq 0$) is called a **hyperbola, which consists of two separate parts.**

From diagram 14-17 and diagram 14-18, we can conclude that **inversely proportional function $y = \frac{k}{x}$ has the following characteristics:**

(1) When $k > 0$, the graph consists of two separate parts , one part lies in the first quadrant and the iother part lies in the third quadrant. In each quadrant, *y* decreases as *x* increases; when $k < 0$ the graph consists of two separate parts, one part lies in the second and the other part lies in the fourth quadrant. In each quadrant, *y* increases as *x* increases.

(2) Both parts will approach to but will never touch the *x axis* and the *y axis*.

Practice

1. (*Mental*) In the following questions, are the two variables inversely proportional? Why?
   (1) Consider the variable of the distance traveled and the variable of the uniform velocity, when an object travels for a fixed period of time under uniform vlocity;
   (2) Consider the varible of the uniform velocity and the varible of the traveling time, when an object travels over a fixed distance.
2. Given that that y and x are inversely proportional and that when $x=3$, $y=7$. Find the following:
   (1) the function of y in terms of x;
   (2) the value of y when $x=2\frac{1}{3}$;
   (3) the value of x when $y=3$.
3. In the same rectangular coordinate plane, draw the following graphs:

$$y=\frac{5}{x}, \quad y=\frac{-5}{x}.$$

Exercise 5

1. In each of the following questions, classify whether the variables are (i) directly proportional (ii) inversely propostional (iii) not proprotioanl
   (1) given that the length of the base line of a triangle is unchanged, consider the variable of area and the variable of height;
   (2) given that the area of a triangle is unchanged, consider the varibable of the length of its base line and the variable of its height;
   (3) given that the weight of an object is unchanged, consider the variable of its volume and the variable of its density;

   (4) given that the volume of an object is unchanged, consider the variable of its weight and the variable of its density;
   (5) the age and weight of a person;
   (6) given that the divisor in a division calculation is unchanged, consider the variable of the dividend and the variable of the quotient;
   (7) given that the dividend in a divison calculation is unchanged, consider the variable of the divisor and the variable of the quotient;
   (8) $x+3$ and x;
   (9) given that $xy=18$, consider the variable of y and the variable of x;
   (10) given that $x \div y=18$, consider the variable of y and the variable of x.
2. Given that y and x are directly proportional and that when $x=8$, $y=6$, find the corresponding values of y
   (i) when $x=3$ and (ii) when $x=-9$.
3. Given that y and x^2 are directly proportional and that when $x=2$, $y=16$, find the following:
   (1) the value of y when $x=-4$;
   (2) the value of x when $y=64$.
4. In the same rectangular coordinate plane, draw the graphs of the following functions:

$$y=\frac{3}{4}x, \quad y=-\frac{5}{2}x, \quad y=2.5x, \quad y=-0.6x.$$

5. Given that a and b^2 are inversely proportional and that when $b=4$, $a=5$, find the value of a when $b=\frac{4}{5}$.
6. The area of a rectangle is 24 cm^2, Its length is x cm.
   (1) Find the width y;
   (2) Write the function of the width in terms of the length of the rectangle. Write down the domain of possible values of its length. Draw the graph of the function.

7. In the same rectangular coordinate plane, draw the graphs of the following functions:
$$xy = 1,\quad xy = -1,\quad xy - 2 = 0,\quad xy + 2 = 0.$$
8. It is known that (i) $y = y_1 + y_2$, (ii) y_1, and x are directly proportional, (iii) y_2 and x^2 are inversely proportional, (iv) when $x = 2$ and also when $x = 3$, the value of y is equal to 19. Find the function for y in terms of x.
9. It is known that (i) $y = y_1 + y_2$, (ii) y_1 and x are directly proportional, (iii) y_2 and x are inversely proportional, (iii) when $x = 1$ and $y = 4$, (iv) when $x = 2$, $y = 5$.
10. Find the value of y when $x = 4$.

IV. The graph of a function of first degree

14.7 Function of first degree

Let us study the following examples:

(1) Having left station *A* for 4 km, a car continues in the same direction and travels at a speed of 40 km/hour for *t* hours. After *t* hours, the car is at a distance of, *s* km from station *A*, and the distance *s* km can be represented by the following function
$$s = 40t + 4.$$
(2) At the start of the drilling process, the gas tank of an excavator has 40 kg of fuel. The escavator burns 6 kg of fuel for each hour in operation. After t hours in operation, the amount of fuel remaining in the gas tank *Q* (kg) can be represented by the following function
$$Q = 40 - 6t = -6t + 40.$$
The two functions above are in the form of $y = kx + b$ ($k \neq 0$).

In general, function of the form $y = kx + b$ is called a **first degree function of** *x*. Here *x* is a variable, *k* and *b* are constants, and $k \neq 0$.

If $b = 0$, then the function $y = kx + b$ becomes $y = kx$, which is a diriectly proportional function studied above. Therefore a directly proportional is only a special case of the general first degree function of the variable.

【Example】 A car travels at an average speed of v_0 km/minute from station *A* via station *B* to station *C*. It is known that, 9 minutes after passing through station *B*, the car is at 10 km from station *A*, After another 15 minutes, the car is at 20 km from station *A*. What distance is the car from station *A* after another 30 minutes?

Solution Let the distance between stations *A* and *B* be s_0 km. After leaving station *B* for *t* minutes, the distance *s* km of the car from station *A* can be represented by the formula known in Physics as
$$s = v_0 t + s_0.$$
Substituting the two known values of $s = 10$ at $t = 9$ and of $s = 20$ at $t = 24$ to the formula, we get
$$\begin{cases} 10 = 9v_0 + s_0 \\ 20 = 24v_0 + s_0 \end{cases}$$
We obtain two simultaneous equations of the first degree in two unknowns.

Solving the simultaneous equations, we get
$$v_0 = \frac{2}{3},\quad s_0 = 4.$$
$$\therefore\quad s = \frac{2}{3}t + 4$$
When $t = 54$, $s = \frac{2}{3} \times 54 + 4 = 40$ (km).

Answer: At 54 minutes after leaving station *B*, the car is 40 km from station *A*.

In the above example

(i) We first establish the relationship of s as represented by the function $s = v_0 t + s_0$, where v_0, s_0 are unknown coefficients.

(ii) Then based on two sets of known values that $s = 10$ at $t = 9$ and that $s = 20$ at $t = 24$, we obtained $v_0 = \frac{2}{3}$, $s_0 = 4$.

This method allows us to establish the function using some unknown coefficients and then solve this equations basing on some given conditions. This method is called **undetermined coefficient method**. This method of solving problem is a common technique used in mathematics

Practice

Given that $y-3$ and x are inversely proportional and that when $x=2$, $y=7$.

(1) Write down the function for y in terms of x;

(2) When $x=4$, what is the value of y;

(3) When $y=4$, what is the value of x.

14.8 Graph of first degree functions

Let us draw the graph of the function $y=\frac{2}{3}x+4$, and compare it with the graph of the function $y=\frac{2}{3}x$.

As usual, we pick a few values for x and fill up the following table:

x	$\cdots$	-2	-1	0	1	2	$\cdots$
$y=\frac{2}{3}x$	$\cdots$	$-\frac{4}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{4}{3}$	$\cdots$
$y=\frac{2}{3}x+4$	$\cdots$	$-\frac{4}{3}+4$	$-\frac{2}{3}+4$	4	$\frac{2}{3}+4$	$\frac{4}{3}+4$	$\cdots$

The graph is drawn in diagram 14-19

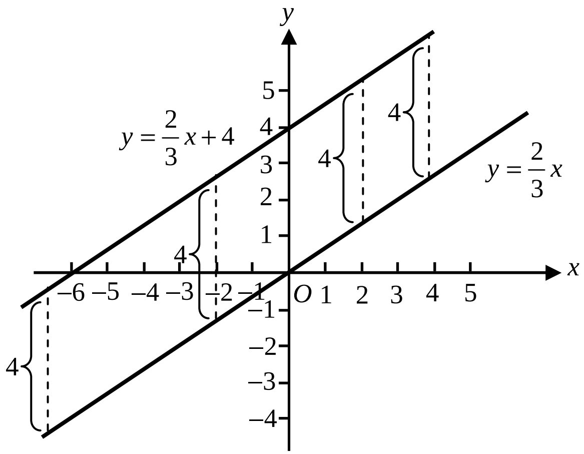


Diagram 14-19

We can observe that, for each value of x, the function $y=\frac{2}{3}x+4$ is greater than the function $y=\frac{2}{3}x$ by 4 units. Therefore, the straight line $y=\frac{2}{3}x$ can be moved upwards by 4 units to form the graph of the function $y=\frac{2}{3}x+4$. So we know that the first degree function $y=\frac{2}{3}x+4$ is parallel to the straight line $y=\frac{2}{3}x$ and passes throught the point (0, 4)

In general, the graph of the first degree function $y=kx+b$ is parallel to the straight line $y=kx$ and passes through the point (0, b). Therefore, we call the graph of the funciton $y=kx+b$ a straight line $y=kx+b$.

The straight line $y=kx+b$ cuts the y-axis at the point B(0, b). Here b is called the intercept of straight line $y=kx+b$ with the y-axis or called in short, **the *y*-intercept**.

A first degree function $y=kx+b$ has the following characteristics:

When $k>0$, y increases as x increases; when $k<0$, y decreases as x increases.

We know that the position of a straight line can be determined by any two points on the line. So, to draw the graph of $y=kx+b$, we shall first need to find any two points on the line, then we can draw a line passes through these two points to form the graph.

【Example 1】 Draw the straight line $y=\frac{1}{2}x+2$.

Solution First calculate two points on the line.

When $x=0$, $y=2$;

when $y=0$, $x=-4$.

Next, draw a straight line passes through the point $A(0, 2)$ and the point $B(-4, 0)$. This is the required line (diagram14-20).

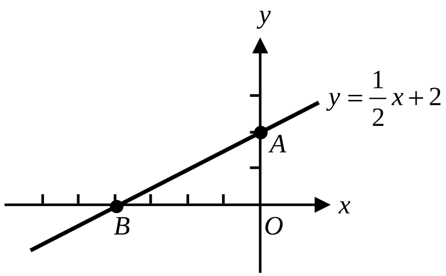


Diagram 14-20

【Example 2】 A spring is in its original length of 12 cm when there is no load of weight attached to it. It can carry a load no more than 15 kg. For every load of 1 kg it carries, its length would increase by $\frac{1}{2}$ cm. Write the function of the length of the spring, y (cm) in terms of the load of weight it carries, x (kg). Draw a graph of the function.

Analysis: As a load of every 1 kg added to the spring, the length of the spring is increased by $\frac{1}{2}$ cm. Therefore a load of x kg added to the spring will increase the length of the spring by $\frac{1}{2}x$ cm. Because the original length of the spring is 12 cm, when a load of weight x kgis attached to it, the length of the spring becomes $\left(\frac{1}{2}x+12\right)$cm.

Solution With the above analysis, we know that the function for y in terms of x is

$$y=\frac{1}{2}x+12 \quad (0\le x\le 15).$$

(The inequality of $0\le x\le 15$ inside the bracket shows the domain of possible values of x.)

Let us draw the line $y=\frac{1}{2}x+12 \quad (0\le x\le 15)$.

When $x=0$, $y=12$; when $x=15$, $y=19\frac{1}{2}$.

Plot the two points $A(0, 12)$ and $B(15, 19\frac{1}{2})$ on the rectangular coordinate plane., Connect the points A and B.

(Think for a while: why don't we extend the line at both ends beyond the points A and B?).

AB is the required graph (refer diagram 14-21).

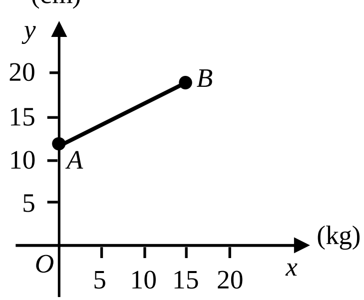


Diagram 14-21

Practice

1. In the same rectangular coordinage plane, draw the graphs of the following functions, and compare them with the graph of the straight line $y = \frac{1}{3}x$.

   (1) $y = \frac{1}{3}x + 4$; (2) $y = \frac{1}{3}x - 2$.

2. (1) Given that a first degree function $y = kx + 2$ has the value 4 at $x = 5$, find the value of *k*;

   (2) Given that the straight line $y = kx + 2$ passes through the point *P*(5, 4), draw this line.

Exercise 6

1. It is known that if pressure remains the same, the relationship of the volume of a gas and its temperature is given by the function $V_t = V_0 + 0.0037V_0 t$, where V_t is the volume of the gas at temperature t °C, V_0 is the volume of the gas at temperature 0 °C. Now we have a certain amount of the gas with pressure kept unchanged. At 0 °C, the volume of the gas is 100 L. Find the following:

   (1) The volume of gas at 30 °C.

   (2) At what temperature will the volume of the gas become 101 L.

2. It is known that $y = P + z$, where *P* is a constant, and *z* is directly proportional to *x*. When $x = 2$, $y = 1$. When $x = 3$, $y = -1$.

   (1) Write down the function of y in terms of *x*;

   (2) Find the value of *y* when $x = 0$;

   (3) Find the value of *x* when $y = 0$.

3. Given that $y + b$ is directly proportional to $x + a$ (where *a* and *b* are constants).,

   (i) Prove that *y* is a first degree function of *x*.

   (ii) When $x = 3$, $y = 5$. When $x = 2$, $y = 2$. Write *y* as a function of *x*.

4. From experiment, it is known that the volume of alcohol is approximately a first degree function with its temperature. AThe volume of a certain amount of alcohol at 0 °C is 5.250 L. Its volume at 40 °C is 5.481 L. Find the volume of the alcohol at 10 °C and the volume of the alchol at 30 °C.

5. Sound travels in air at a speed of *v* (m/second) which is related to the air temperatue *t* (°C) by the function $v = 331 + 0.6t$. Draw the graph of this function. From the graph, find the speed of sound (i) when $t = -5$ °C and (ii) when $t = 15$ °C.

6. (1) Given that a first degree function $y = kx + b$ has the value of 9 when $x = -4$, and has the voalue of 3 when $x = 6$. Find the values of *k* and *b*;

   (2) Given that the straight line $y = kx + b$ passes through the point (−4, 9) and the point (6, 3), find the values of *k* and *b*. Draw the straight line.

7. (1) In the same rectangular coordinate plane, draw the graphs of the following lines:

   $$y = 2x + 3,\quad y = 2x - 3,\quad y = -x + 3,\quad y = -x - 3.$$

   (2) Do these four lines from a parallelogram? Why?

8. (1) In the same coordinate plane, draw the graphs of the function $y = 3x - 2$ and the function $y = 2x + 3$;

   (2) From these graphs, find the value of *x* at which the function $y = 3x - 2$ and the function $y = 2x + 3$ have the same value;

   (3) In a similar manner, draw graphs to solve the equation $6x + 3 = 4x - 7$.

9. If the graph $y=kx+b$ falls in the quadrants specified below, sketch the graph and identify the signs of k and b.
   (1) in the first, second and third quadrants;
   (2) in the first second and fourth quadrants;
   (3) in the first, third and fourth quadrants;
   (4) in the second, third, and fourth quadrants.
10. Draw the graph of the function $y=3x+12$. From the graph:
   (1) find the corresponding values of y, when $x=-2$, -1, $\frac{1}{2}$ respectively.
   (2) find the corresponding values of x hen $y=3$, 9, -3 respectively.
   (3) find the coordinates of the two points where the graph cuts the x-axis and the y-axis, and also the distance between these two points;
   (4) solve the equation $3x+12=0$;
   (5) solve the inequality $3x+12>0$;
   (6) if y has a value between $-6\le y\le 6$, find the domain of possible values of x.

V. Graph of function of second degree

14.9 Function of second degree

Let us study the following examples:

(1) The side of a square is x (cm), then its area y is related to its side x by this function

$$y=x^2\ (\text{cm}^2)$$

(2) A factory produces 50 sets of TV in the first month. Its production rate increases from month to month as a steady rate x(%). The production in the third month y is related to x by the following function

$$y=50(1+x)^2,$$

That is,

$$y=50x^2+100x+50.$$

In the above function, the independent variable x has the highest power of 2. The function with a general form $y=ax^2+bx+c$ (where a, b, c are constants, and $a\ne 0$) is called a **function of second degree**.

Practice

1. A rectangular wooden plank measures a cm long and b cm wide. If both the length and the width are shortened by x cm. find the function that relates the area of the plank y (cm^2) to x (cm).
2. The height of a cylinder h (cm) is of constant value. The circumference of the circular base is C (cm). Find the function which expresses the volume of the cylinder V (cm^3) in terms of the circumference of the circular base of C (cm).

14.10 Characteristic of graph of function $y=ax^2$

Let us first study the characteristics of some specific graphs of second degree functions. After that, we shall study the characteristics of second degree functions in general.

【**Example 1**】Draw the graphs of the function $y=x^2$ and of the function $y=-x^2$.

Solution From the domain of possible values of x, pick some values for x, calculate the corresponding values for function y and fill in the following table:

x	$\cdots$	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$\cdots$
$y=x^2$	$\cdots$	4	$2\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	$2\frac{1}{4}$	4	$\cdots$

Regard each pair of data as the coordinates of points on the rectangular corrdinate plane and plot the points there. Connect these points smoothly, and it will form the graph of the function $y = x^2$ (refer diagram14-22).

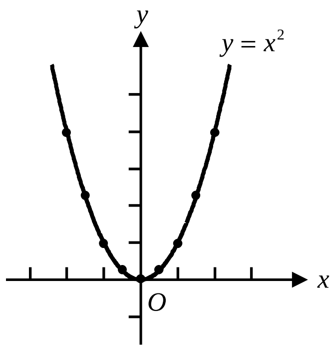


Diagram 14-22

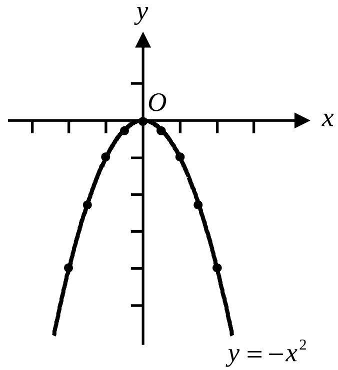


Diagram 14-23

In a similar maner, draw the graph of function $y = -x^2$ (refer diagram 14-23).

【Example 2】 Draw the graphs for the function $y = \frac{1}{2}x^2$ and the function $y = 2x^2$.

Solution We first draw the graph for the function $y = \frac{1}{2}x^2$. From the domain of possible values of *x*, pick some values for *x*, calculate the corresponding values for the function *y* and fill in the following table:

x	$\cdots$	-4	-3	-2	-1	0	1	2	3	4	$\cdots$
$y = x^2$	$\cdots$	8	$4\frac{1}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$4\frac{1}{2}$	8	$\cdots$

Regard each pair of the data as a point on the rectangular coordinate plane, and plot the points there. Connect the points by a smooth line, and it will form the graph of the function $y = \frac{1}{2}x^2$ (Refer diagram 14-24).

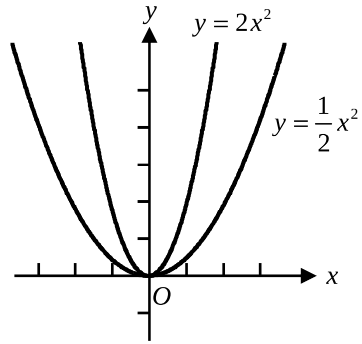


Diagram 14-24

In a similar manner, draw the graph of the function $y = 2x^2$ (refer diagram 14-24).

The graph of function $y = ax^2$ is like the trajectory of an object being thrown into the air and fell back to the ground. The shape of the trajectory is called a **parabola** (though it is turned upside down).**. The** parabola is symmetric on both sides of the *y* axis. The line of symmtry (in this case, it is the *y* axis*)* is called the **symmtry line.** The point of intersection between the parabola and the symmtry line is called the **vertex. In this particular case, the vertex of the parabola is the** origin.

From the diagrams 14-22, 14-23, 14-24, we observe that second degree function of the form $\boldsymbol{y = ax^2}$ **has the following characteristics:**

(1) the vertex of parabola $\boldsymbol{y = ax^2}$ is the origin, and the symmtry line is the *y- axis.*

(2) when $\boldsymbol{a > 0}$, the parabola $\boldsymbol{y = ax^2}$ lies above the *x* axis (vertex is on the *x axis*). Its month opens upward, and extends upward to infinitely;
when $\boldsymbol{a < 0}$, the parabola $\boldsymbol{y = ax^2}$ lies below the *x* axis (vertex is on the *x* axis). Its mouth opens downward, and extends downward to infinity.

(3) **When $a > 0$, on the left side of the symmetric axis, *y* decreases as *x* increases; on the right side of the symmetric axis, *y* increases as *x* increases. Function *y* is at its minimum when $x = 0$.**
When $a < 0$, on the left side of the symmetric axis, *y* increases as *x* increases; on the right side of the symmetric axis, *y* decreases as *x* increases. Function *y* is at tis maximum when $x = 0$.

Practice

1. In the same rectangular coordinate plane, draw the graphs of the following functions and compare their positions:
   (1) $y=\frac{2}{3}x^2$;   (2) $y=-\frac{2}{3}x^2$.
2. The area of a circle is given by the forrmula $A=\pi r^2$, where *r* is the radius, and *A* is the area. Take the value of π as 3.14.
   (1) For $r=3$, 5, 2.5 (cm), respectively, find the corresponding values of the area of the circle;
   (2) Draw the graph of the function $A=\pi r^2$ ($0<r\le 8$);
   (3) From the diagram, find the corresponding values of its radius when $A=20$, 40, 60 (cm^2) respectively.

14.11 Characteristic of graph of function $y=ax^2+bx+c$

【Example 1】 In the same coordinate plane, draw the graphs of the following functions

$$y=\frac{1}{2}x^2$$

$$y=\frac{1}{2}(x+3)^2$$

$$y=\frac{1}{2}(x+3)^2-2$$

Solution From the domain of possible values of *x*, pick some values of *x*, calculate the corresponding values of the functions and fill in the following table:

x	$\cdots$	–6	–5	–4	–3	–2	–1	0	1	2	3	$\cdots$
$y=\frac{1}{2}x^2$				$\cdots$	$4\frac{1}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$4\frac{1}{2}$	$\cdots$
$y=\frac{1}{2}(x+3)^2$	$\cdots$	$4\frac{1}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$4\frac{1}{2}$	$\cdots$			
$y=\frac{1}{2}(x+3)^2-2$	$\cdots$	$2\frac{1}{2}$	0	$-1\frac{1}{2}$	-2	$-1\frac{1}{2}$	0	$2\frac{1}{2}$	$\cdots$			

Plot the points and connect them to obtain the graphs (refer diagram 14-25).

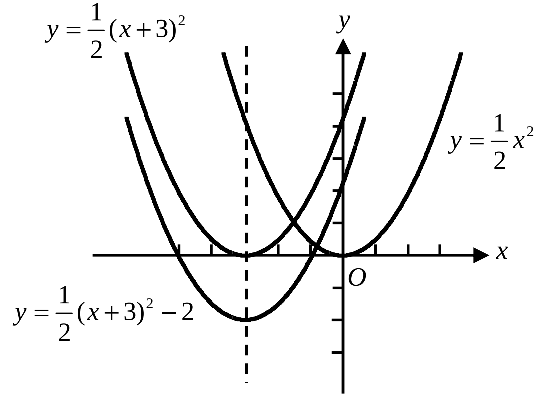


Diagram 14-25

From the diagrams, we observe that,

(i) if the graph of the function $y=\frac{1}{2}x^2$ is moved to the left by 3 units, it becomes the graph of $y=\frac{1}{2}(x+3)^2$.

(ii) if the graph of $y=\frac{1}{2}(x+3)^2$ is moved down by 2 units, it becomes the graph of $y=\frac{1}{2}(x+3)^2-2$.

(iii) Because $\frac{1}{2}(x+3)^2-2=\frac{1}{2}x^2+3x+\frac{5}{2}$, so it is the graph of $y=\frac{1}{2}x^2+3x+\frac{5}{2}$.

From this we conclude that, the graph of function $y=\frac{1}{2}x^2+3x+\frac{5}{2}$ and of function $y=\frac{1}{2}x^2$ have the same shape, except that they are located at different positions on the rectangular coordinate plane. Obviously, the function $y=\frac{1}{2}x^2+3x+\frac{5}{2}$ $=\frac{1}{2}(x+3)^2-2$ is smallest when $x=-3$ and the smallest value is -2. Therefore, the parabola $y=\frac{1}{2}x^2+3x+\frac{5}{2}$ has a vertex at $(-3,-2)$, and the symmetry line passes through $(-3,-2)$ and is parallel to the y axis. Thse symmtry line is the line $x=-3$.[2]

In general, the graph of function $y=ax^2+bx+c$ has the same shape as the graph of function $y=ax^2$, only that it is located at different position in the rectangular coordinate plane. Because

$$
\begin{aligned}
y&=ax^2+bx+c\\
&=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)\\
&=a\left[x^2+2\bullet\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right]\\
&=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}
\end{aligned}
$$

[2] 「Straight line $x=-3$ 」: means a straight line made up of all points with x-coordinates equal to -3. The line is parallel to the y -axis. In general, 「straight line $x=h$ 」 means a straight line parallel to the y-axis and cutting the x-axis at x = h.

The graph can be obtained by moving the graph of $y=ax^2$:

(i) when $\frac{b}{2a}>0$, move the graph of $y=ax^2$ to the left by $\frac{b}{2a}$ units;

(ii) when $\frac{b}{2a}<0$, move the graph of $y=ax^2$ to the right by $\left|\frac{b}{2a}\right|$ units;

(iii) when $\frac{4ac-b^2}{4a}>0$, move the graph of $y=ax^2$ up by $\frac{4ac-b^2}{4a}$ units;

(iv) when $\frac{4ac-b^2}{4a}<0$, move the graph of $y=ax^2$ down by $\left|\frac{4ac-b^2}{4a}\right|$ units.

We further note the following points:

(1) The graph of function $y=ax^2+bx+c$ is a parabola, and the vertex is located at the point $\left(-\frac{b}{2a},\frac{4ac-b^2}{4a}\right)$, the symmetry line is $x=-\frac{b}{2a}$ which is parallel to y axis.

(2) when $a>0$, the parabola $y=ax^2+bx+c$ has its mouth opened upward. Its vertex is the minimum poin. So when $a>0$ and $x=-\frac{b}{2a}$, y has the minimum value, which is

$$y_{\min}=\frac{4ac-b^2}{4a};$$

(3) when $a<0$, the parabola $y=ax^2+bx+c$ has its mouth opened downward. Its vertex is the maximum poin. So when $a<0$ and $x=-\dfrac{b}{2a}$, y has the maximum value, which is

$$y_{\max}=\frac{4ac-b^2}{4a}.$$

Summing up the above, a **second degree function $y=ax^2+bx+c$ has the following characteristics:**

(1) the vertex of the parabola $y=ax^2+bx+c$ is located at $\left(-\dfrac{b}{2a},\dfrac{4ac-b^2}{4a}\right)$. The symmtry line is the straight line $x=-\dfrac{b}{2a}$.

(2) (i) when $a>0$, the graph opens upward infinitely;

(ii) when $a<0$, the graph opens downward infinitely.

(3) (i) when $a>0$, on the left side of the symmtry line, y decreases as x increases; on the right side of the, y increases as x increases; function y is minimum when $x=-\dfrac{b}{2a}$ and its minimum value is $\dfrac{4ac-b^2}{4a}$.

(ii) when $a<0$, on the left side of the symmtry line, y increases as x increases; on the right side of the symmetric axis, y decreases as x increases; function y is maximum when $x=-\dfrac{b}{2a}$ and its maximum value is $\dfrac{4ac-b^2}{4a}$.

【**Example 2**】Fnd the symmtry line and vertex of the parabola

$y=-\dfrac{1}{2}x^2-3x-\dfrac{5}{2}$ and draw the graph.

Solution In function $y=-\dfrac{1}{2}x^2-3x-\dfrac{5}{2}$, $a=-\dfrac{1}{2}$, $b=-3$, $c=-\dfrac{5}{2}$, therefore

$$-\frac{b}{2a}=-3,\quad \frac{4ac-b^2}{4a}=2.$$

Another approach is to use the completing square method for $-\dfrac{1}{2}x^2-3x-\dfrac{5}{2}$, then the function can be re-grouped into

$$y=-\frac{1}{2}(x^2+6x+5)=-\frac{1}{2}[(x+3)^2-4]=-\frac{1}{2}(x+3)^2+2.$$

Therefore parabola $y=-\dfrac{1}{2}x^2-3x-\dfrac{5}{2}$ has symmtry line $x=-3$ and vertex $(-3, 2)$.

From the domain of possible values of x, pick some values of *x*, calculate the corresponding values of the function and obtain the following table:

x	$\cdots$	-6	-5	-4	-3	-2	-1	0	$\cdots$
y	$\cdots$	$-2\frac{1}{2}$	0	$1\frac{1}{2}$	2	$1\frac{1}{2}$	0	$-2\frac{1}{2}$	$\cdots$

Plot the points and connect them to obtain the graph (refer diagram 14-26).

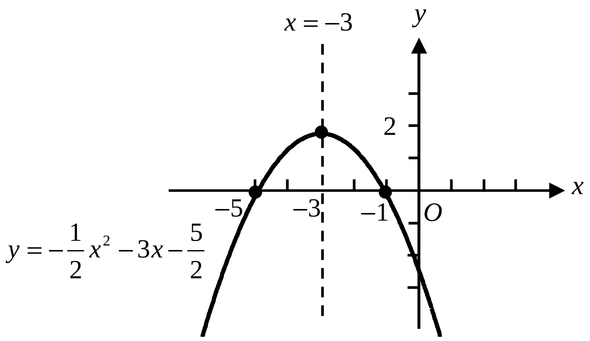


Diagram 14-26

【**Example 3**】For the second degree function $y=2x^2-8x+1$, find its maximum or minimum value.

Solution $y=2x^2-8x+1=2(x^2-4x+4)-8+1=2(x-2)^2-7$,

Because $a=2>0$, therefore y has a minimum value. When $x=2$,

$$y_{\min}=-7.$$

Practice

1. Use allocation method to re-write the following function in the form of $y=a(x+h)^2+k$. Find the symmtry line, the vertex and identify whether the graph opens upward or downward:

(1) $y=x^2-2x-3$; (2) $y=x^2+6x+10$;

(3) $y=2x^2-3x+4$; (4) $y=-2x^2-5x+7$;

(5) $y=3x^2+2x$; (6) $y=\frac{5}{2}x-2-3x^2$.

2. Draw draw the graph of the following function:

(1) $y=-x^2-2x$; (2) $y=1-3x^2$;

(3) $y=-2x^2+8x-8$; (4) $y=\frac{1}{2}x^2+3x+\frac{5}{2}$.

3. (1) Find the values of the function $y=2(x-3)^2$ when $x=1$, 2, 2.5, 2.9, 3, 3.1, 3.5, 4, 5 respectively. At which of these points, the value of the function is the smallest? What is the smallest value of the function?

(2) Find the values of the function $y=4-(x+2)^2$ when $x=-5$, -4, -3, -2, -1, 0, 1 respectively. At which of these points, the value of the function is the largest? What is the largest value of the function?

4. Find the maximum or the minimum of the following function:

(1) $y=x^2-2x+4$; (2) $y=-x^2+3x$;

(3) $S=1-2t-t^2$; (4) $u=2V^2+4V-5$;

(5) $V=-3t^2+4t$; (6) $y=x(8-x)$;

(7) $h=100-5t^2$; (8) $y=(x-2)(2x+1)$.

Exercise 7

1. (1) Let the flow speed be a m/minute, find the function which relates the volume of water V (m^3) flowing through a pipe and the diameter D (m) of the pipe;

(2) the length of a side of a square is 3. If the length of the side is increased by x, then its area is increased by y. Find the function which relates y and x.

2. Draw the graph of the function $y=x^2$ and from the graph find the following:

(1) when $x=2$, 2.4, -1.7 respectively, fomd the corresponding values of y (accurate to 0.1);

(2) when $x=1.2^2$, $(-2.3)^2$ respectively, find the corresponding values of y (accurate to 0.1);

(3) when $y=2$, 5.8 respectively, find the corresponding values of x (accurate to 0.1);

(4) when $y=\sqrt{3}$, $\sqrt{8}$ respectively, find the corresponding values of x (accurate to 0.1).

3. For a second degree function in x, its value is 4 when $x=0$; its value is 3 when $x=1$; its value is 6 when $x=2$. Find this function.

4. A parabola $y=ax^2+bx+c$ passes through the three points $A(0, 1)$, $B(1, 3)$, $C(-1, 1)$. Find the value of a, b, c and draw the graph of this parabola.

5. For the functions $y=\frac{1}{4}x^2$, $y=-\frac{3}{2}x^2+2$, $y=x^2+\frac{1}{2}x$, $y=3x^2-4x+1$.

(1) Draw their graphs.

(2) Find their symmtry line, the coordinate of the vertex and the opening direction.

(3) From the graph, what values of x will enable the value of the function

(i) to be greater than zero; (ii) to be less than zero;
(iii) to be equal to zero.

VI. Simultaneous Linear and Quadratic Inequalities in one variable

14.12 Simultaneous linear inequalities in one variable

We have learnt linear inequality in one varable before. Now we move on to study simultaneous linear inequalies in one variable.

A number of linear inequalities in one variable operating together constitute a group of simultaneous linear inequalities in one variable. Each of the constituent inequality has its solution set. A solution set common to all the solution sets of the constituent inequalities is the solution set of the group of simultaneous linear inqualities.

【**Example 1**】Solve this set of inequalities

$$\begin{cases} 2x-1>x+1 \\ x+8<4x-1 \end{cases}$$

Solution The solution set of inequality $2x-1>x+1$ is $x>2$.
The solution set of inequality $x+8<4x-1$ is $x>3$.
Therefore the solution set for this group of simultaneous inequalities is $x>3$.
Graphical representation of the solution set is shown in diagram 14-27.

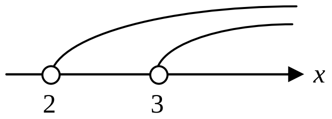


Diagram 14-27

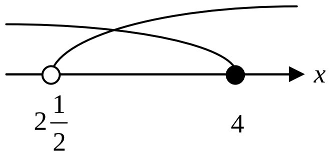


Diagram 14-28

【**Example 2**】Solve this group of inequalities

$$\begin{cases} 5x-2>3(x+1) \\ \dfrac{1}{2}x-1\le 7-\dfrac{3}{2}x \end{cases}$$

Solution The solution set for the inequality $5x-2>3(x+1)$ is $x>2\frac{1}{2}$.
The solution set for inequality $\frac{1}{2}x-1\le 7-\frac{3}{2}x$ is $x\le 4$.
Therefore the solution set for this group of simultaneous inequalities is $2\frac{1}{2}<x\le 4$.
Graphical representation of this solution set is shown in diagram 14-28.

【**Example 3**】Solve this group of simultaneous inequalities

$$\begin{cases} 2x+3<5 \\ 3x-2>4 \end{cases}$$

Solution The solution set for the inequality $2x+3<5$ is $x<1$.
The solution set for the inequality $3x-2>4$ is $x>2$.
From this we know that there is no common solution to both solution sets. In other words, there is no number that can satisfy both inequalities. Therefore the solution set is an empty set.

From the above, we know that the solution set to a group of simultaneous linear inequalities in one variable has the four different situation .

Let $a<b$, then:

1. for the group of simultaneous inequalities

$$\begin{cases} x>a \\ x>b \end{cases}$$

The solution set is $x>b$ (refer diagram14-29);

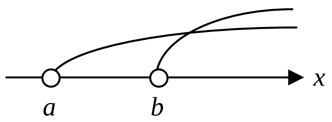


Diagram 14-29

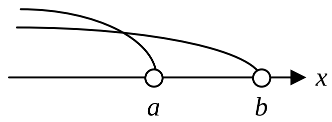


Diagram 14-30

2. for the group of simultaneous inequalities

$$\begin{cases} x < a \\ x < b \end{cases}$$

The solution set is $x < a$ (refer diagram14-30);

3. for the group of simultaneous inequalities

$$\begin{cases} x > a \\ x < b \end{cases}$$

The solution set is $a < x < b$ (refer diagram14-31);

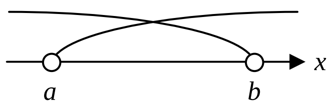


Diagram 14-31

4. for the group of simultaneous inequalities

$$\begin{cases} x < a \\ x > b \end{cases}$$

The solution set is empty.

Practice

Solve the following inequality group. If the solution set is non empty, draw it on the number axis:

(1) $\begin{cases} x > -4 \\ x < 2 \end{cases}$ (2) $\begin{cases} x > -5 \\ x > -3 \end{cases}$ (3) $\begin{cases} x < 7 \\ x < -1 \end{cases}$

(4) $\begin{cases} x < 0 \\ x > 3 \end{cases}$ (5) $\begin{cases} 2x - 1 > 0 \\ 4 - x > 0 \end{cases}$ (6) $\begin{cases} -3x < 0 \\ 4x + 7 > 0 \end{cases}$

14.13 Inequality of the form $|x| < a$, $|x| > a$ $(a > 0)$ and its solution

Find the solution set for $|x| < a$, $|x| > a$ $(a > 0)$.

Example, solve $|x| < 2$.

From the definition of absolute value we know that, $|x| < 2$ can be re-writem as a group of the following simultaneous inequalities:

$$\begin{cases} x \ge 0 \\ x < 2 \end{cases} \quad (1)$$

or

$$\begin{cases} x < 0 \\ -x < 2 \end{cases} \quad (2)$$

The solution set for group (1) is $0 \le x < 2$.

The solution set for (2) is $-2 < x < 0$.

Therefore the solution set for $|x| < 2$ is $-2 < x < 2$ (refer diagram14-32).

From diagram14-32, we know that the solution set for $|x| < 2$ are the points within 2 units from the origin (except the boundary points).

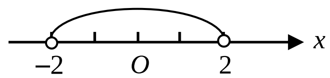


Diagram 14-32

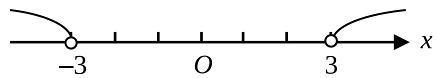


Diagram 14-33

Solve the inequality $|x| > 3$.

The inequality can be split into a group of the following simultaneous inequalities:

$$\begin{cases} x \ge 0 \\ x > 3 \end{cases} \quad (1)$$

or

$$\begin{cases} x < 0 \\ -x > 3 \end{cases} \quad (2)$$

The solution set for group (1) is $x > 3$.

The solution set for group (2) is $x < -3$.

Therefore the solution set for, $|x| > 3$ is $x > 3$ or $x < -3$ (refer diagram14-33).

From diagram 14-33, we know that the solution set for $|x| > 3$ are the points more than 3 units from the origin (except the two boundary points).

In general, the solution set for inequality $|x| < a$ ($a > 0$) is $-a < x < a$ (refer diagrm 14-34).; the solution set for inequality $|x| > a$ ($a > 0$) is $x > a$ or $x < -a$ (refer diagram 14-35).

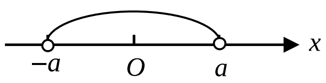


Diagram 14-34

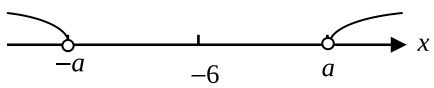


Diagram 14-35

【**Example 1**】Solve the inequality $|3x| < 8$.

Solution From the previous example we know that

$$-8 < 3x < 8.$$

Dividing both sides by 3, we get

$$-2\frac{2}{3} < x < 2\frac{2}{3}.$$

The solution set is $-2\frac{2}{3} < x < 2\frac{2}{3}$.

【**Example 2**】Solve the inequality $|x-5| < 8$.

Solution From the previous example, we know that

$$-8 < x-5 < 8.$$

Subtract 9 from both sides, we get

$$-3 < x < 13.$$

The solution set is $-3 < x < 13$.

【**Example 3**】Solve the inequality $|x+9| \le 86$.

Solution From the previous example, we know that

$$-86 \le x+9 \le 86.$$

Subtract 9 from both sides, we get

$$-95 \le x \le 77.$$

The solution set is $-95 \le x \le 77$.

【**Example 4**】Solve the inequality $|x-3| > 5$.

Solution From the previous example, we know that

$$x-3 > 5 \quad \text{or} \quad x-3 < -5.$$

That is

$$x > 8 \quad \text{or} \quad x < -2.$$

The solution set is $x > 8$ or $x < -2$.

【**Example 5**】Solve the inequality $|x+6| \ge 53$.

Solution From the previous example, we know that

$$x+6 \ge 53 \quad \text{or} \quad x+6 \le -53.$$

In other words

$$x \ge 47 \quad \text{or} \quad x \le -59.$$

Therefore the solution set is $x \ge 47$ or $x \le -59$.

Practice

1. Solve the following inequalities. If the solution set is non empty, draw the solution set on the number axis:

   (1) $|x| < 4$; (2) $|x| > 4$.

2. Solve the following inequalities:

   (1) $|x+4| > 9$; (2) $\left|\frac{1}{4}+x\right| \le \frac{1}{2}$;

   (3) $|2-x| \ge 3$; (4) $\left|x-\frac{2}{3}\right| < \frac{1}{3}$.

14.14 Quadratic inequality in one varable

If an inequality consists of one variable and the highest degree of the variable is two, then the inequality is called a **quadratic inequality in one varable. Its** general form is

$$ax^2+bx+c > 0 \quad \text{or} \quad ax^2+bx+c < 0 \quad (a \ne 0).$$

We shall use our knowledge of the graph of second degree function to discuss the solution to quadratic inequality in one varable.

For example, for the second degree function $y = x^2 - x - 6$, we shall find

(1) what values of x will result in $y = 0$;

(2) what values of x will result in $y > 0$;

(3) what alues of x will result in $y < 0$.

Draw the graph of parabola $y = x^2 - x - 6$. As seen in diagram 14-36 the graph and the x axis intercept at $(-2, 0)$ and at $(3, 0)$. These two points divide the x axis into 3 sections. From the graph (refer diagram 14-36), we can answer the three questions posted above:

(1) when $x = -2$, or $x = 3$, $y = 0$;

(2) when $x < -2$, or $x > 3$, $y > 0$;

(3) when $-2 < x < 3$, $y < 0$.

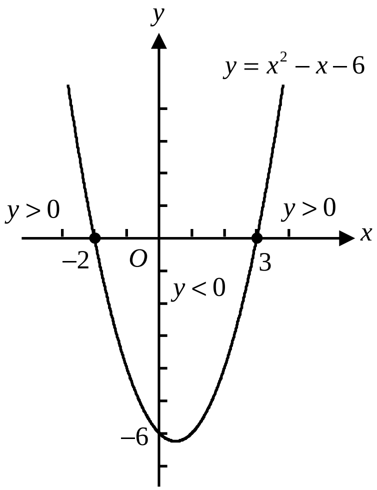


Diagram 14-36

So parabola $y = x^2 - x - 6$ intercepts the x *axis* at two points, which are the real roots of the equation $x^2 - x - 6 = 0$. Equation has two real roots (unequal real roots) $x_1 = -2$, $x_2 = 3$. In this case, the solution set of the inequality $x^2 - x - 6 > 0$ is

$$x < -2, \text{ or } x > 3;$$

The solution set for the inequality $x^2 - x - 6 < 0$ is

$$-2 < x < 3.$$

In general, for the second degree function $y = ax^2 + bx + c$ ($a > 0$), we construct another a function (called the discriminant) with formula $\Delta = b^2 - 4ac$ and calculate its value

1. if $\Delta > 0$, then the parabola $y = ax^2 + bx + c$ and x axis have two interception points (refer diagram 14-37). The two intersection points are the real roots of $ax^2 + bx + c = 0$, which we denote by x_1 and x_2 ($x_1 < x_2$). Then the solution set to the inequality $ax^2 + bx + c > 0$ is

$$x < x_1, \text{ or } x > x_2;$$

The solution set to the inequality $ax^2 + bx + c < 0$ is

$$x_1 < x < x_2.$$

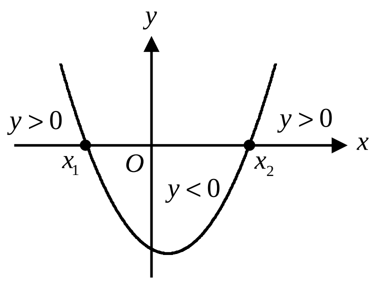


Diagram 14-37

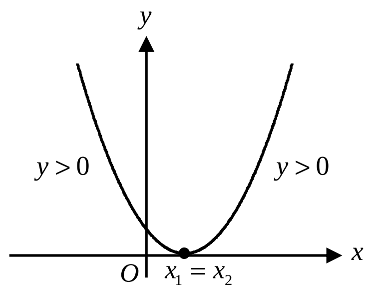


Diagram 14-38

2. if $\Delta = 0$, $y = ax^2 + bx + c$ and the x axis have only one interception point (refer diagram14-38), That means the equation $ax^2 + bx + c = 0$ has two equal roots, $x_1 = x_2 = -\frac{b}{2a}$. Then the solution set for the inequality $ax^2 + bx + c > 0$ consists of all real numbers not equal to $-\frac{b}{2a}$ and the solution set to the inequality $ax^2 + bx + c < 0$ is empty.

3. if $\Delta < 0$, then the parabola $y = ax^2 + bx + c$ and the x axis have no interception point (refer diagram 14-39). That means the equation $ax^2 + bx + c = 0$ has no real

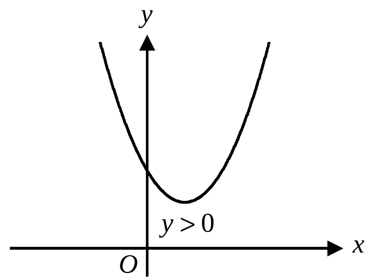


Diagram 14-39

root. Then the solution set of the inequality $ax^2+bx+c>0$ consist of all real numbers, and the solution set of the inequality $ax^2+bx+c<0$ is an empty set.

If the coefficient of the second degree variable is negative (that is $a<0$), we can transpose all the terms to the other side of the inequality sign, in which ase the coefficient of the second degree variable will become positive (that is $-a$ becomes the coeffient, and $-a>0$). Then we can use the above method to find the solution set.

【Example 1】 Solve the inequality $(x+4)(x-1)<0$.

Solution In the expansion of $(x+4)(x-1)$, it is noted that the coefficient ot the second degree variable is positive, so the shape of the graph opens upward infinitely. It is also noted that the roots of $(x+4)(x-1)=0$ are

$$x_1=-4,\ \ x_2=1.$$

Therefore the solution set is

$$-4<x<1.$$

【Example 2】 Solve the inequality $2x^2-3x-2>0$.

Solution Because $\Delta=b^2-4ac>0$, there are two roots in the equation. Solving the equation $2x^2-3x-2=0$, the roots are

$$x_1=-\frac{1}{2},\ \ x_2=2.$$

Therefore the solution of the equality is

$$x<-\frac{1}{2}\ \text{ or }\ x>2.$$

【Example 3】 Solve the inequality $-3x^2+6x>2$.

Solution Transpoing all terms to the right side, we can re-write the inequality as

$$3x^2-6x+2<0.$$

Because $\Delta>0$, there are two real roots in the equation. Solving the equation $3x^2-6x+2=0$, the roots are

$$x_1=1-\frac{\sqrt{3}}{3},\ \ x_2=1+\frac{\sqrt{3}}{3}.$$

Therefore the solution set of the inequality is

$$1-\frac{\sqrt{3}}{3}<x<1+\frac{\sqrt{3}}{3}.$$

【Example 4】 Solve the inequality $4x^2-4x+1>0$.

Solution Because $\Delta=0$, there are two equal roots in the equation. Solving the equation $4x^2-4x+1=0$, the two equal roots are

$$x_1=x_2=\frac{1}{2}.$$

Therefore the solution set of the inequality is any real number not equal to $\frac{1}{2}$.

【Example 5】 Solve the inequality $-x^2+2x-3>0$.

Solution Multiplying both sides by -1, we get

$$x^2-2x+3<0.$$

Because $\Delta<0$, there is no real root in the equation $x^2-2x+3=0$. Hence the solution set of $x^2-2x+3<0$ is empty. So the solution set of the original inequality $-x^2+2x-3>0$ is empty.

【Example 6】 What value of m would enable the equation $x^2-(m+2)x+4=0$ to have real roots?

Solution The discriminant of the equation $\Delta=b^2-4ac$ is

$$[-(m+2)]^2-4\times1\times4=m^2+4m-12.$$

We know that, if the discriminant is greater than or equal to zero, then the equation has real roots.

Solving the equation $m^2+4m-12=0$, we get $m=2$ or $m=-6$;

Hence, we know that for $m^2+4m-12>0$, it is required that $m<-6$ or $m>2$.
Therefore when $m\le -6$ or $m\ge 2$, the original equation has real roots.

Practice

1. Solve the following inequality:
   (1) $(x+2)(x-3)>0$; (2) $x(x-2)<0$;
   (3) $3x^2-7x+2<0$; (4) $4x^2+4x+1<0$;
   (5) $-6x^2-x+2\le 0$; (6) $x(x-1)<x(2x-3)+2$;
   (7) $x^2+10\ge 6x+1$; (8) $x^2-4\frac{1}{3}x+5\frac{1}{3}\le 0$.
2. What value(s) of x would enable the function $y=x^2-4x+1$ to take on the value of:
   (1) zero? (2) a positive number? (3) a negative number?
3. What value(s) of x would enable the function of $\sqrt{x^2+x-12}$ to be meaningful?
4. What value(s) of k would enable the following equation to have real roots?
$$x^2+2x-11=k(3-x)$$

Exercise 8

1. Solve the following simultaneous inequalities:
   (1) $\begin{cases}5x+6>4x\\15-9x<10-4x\end{cases}$; (2) $\begin{cases}x-3(x-2)\ge 4\\\frac{1+2x}{3}>x-1\end{cases}$;
   (3) $\begin{cases}\frac{x}{2}<\frac{x+1}{5}\\\frac{2x-1}{5}<\frac{x+1}{2}\end{cases}$; (4) $\begin{cases}x+2>0\\x-4>0\\x-6<0\end{cases}$.

2. Solve the following simultaneous inequalities:
   (1) $\left|\frac{1}{3}x\right|\ge 7$; (2) $|10x|<\frac{22}{7}$;
   (3) $|x-6|<0.001$; (4) $3\le |8-x|$.
3. Solve the following simultaneous inequalities:
   (1) $4x^2-4x>15$; (2) $14-4x^2\ge x$;
   (3) $x(x+2)<x(3-x)+1$.
4. What values of *xz* would enable the following functions to take on values which are (i) greater than zero? (ii) equal to zero? (iii) less than zero?
   (1) $y=25-x^2$; (2) $y=x^2-14x+45$;
   (3) $y=x^2+6x+10$; (4) $y=-x^2+4x-4$.
5. Find the domain of possible values of x for the following function:
   (1) $y=\frac{1}{\sqrt{x-2}}$; (2) $y=\sqrt{x^2-4}$;
   (3) $y=\sqrt{x+2}\sqrt{x-2}$; (4) $y=\sqrt{-x^2+2x-1}$.
6. (1) What value(s) of m would enable the following equation to have different real roots?
$$x^2+2(m-1)x+3m^2=11$$
   (2) What value(s) of m ($m\ne 0$) would enable the following equation to have real roots?
$$mx^2-(1-m)x+m=0$$
7. Prove that, the following equation will have real roots for any value of k
$$x^2-(k+1)x+k=0.$$
8. What value(s) of k would enable the following simultaneous equations to have a real number solution?
$$\begin{cases}x^2+y^2=16\\x-y=k\end{cases}$$

Chapter Summary

I. This chapter teaches about functions and graphs, and covers the following topics: (i) plane rectangular coordinate system and distance between two points, (ii) concept and representation of functions, (iii) directly proportional functions, inversely proportional functions, (iv) first degree functions and second degree functions, their graphs and characteristics, (v) linear inequality in one variable (unknown), quadratic inequalities in one variable and their solution methods.

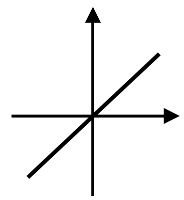
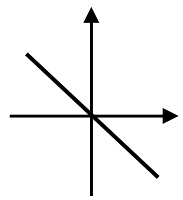
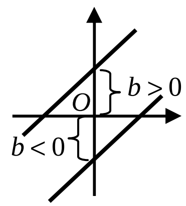
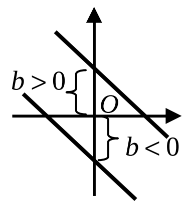
II. In plane rectangular coordinate system, every point on the plane corresponds to a pair of real numbers. There is a one-to-one correspondence between a point on a plane and an ordered pair of real numbers. If we let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be two points on a coordinate plane, then the distance between points P_1 and P_2 is given by the formula

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

III. Observing objects in the world, the motion of all objects can be classified in two different states, namely a relatively static state and a relatively dynamic state. In algebra, the two states correspond to the concept of an unknown value being a constant or a variable. Whether an unknown is a constant or a variable depends on the circumstance, When circumstance changes, the state of an unknown being a constant or a variable may interchange.

IV. Not all variables change independently. Some variable may change interactively with other variable. The inter-relationship between variables may be depicted by a function. The functions tells how the change of one variable may affect the value of the other variable. There are different ways to represent a function, the most common ones being the analytical method, the tabular method and the graphical method.

V. The first degree function $y = kx + b\ (k \neq 0)$ is the simplest function. When $b = 0$, the first degree function becomes a directly proportional funciton $y = kx$ / Its shape and characteristics are as follows:

First degree function $y = kx + b\ (k \neq 0)$			
$b = 0$ (function becomes $y = kx$)		$b \neq 0$	
$k > 0$	$k < 0$	$k > 0$	$k < 0$
			

VI. The graph of inversely proportional function $y = \frac{k}{x}\ (k \neq 0)$ contists of two separate curves.

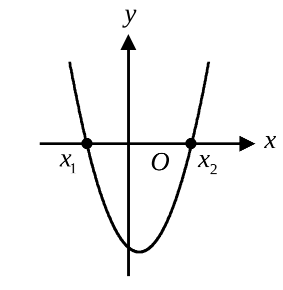
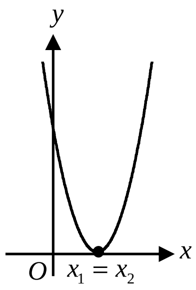
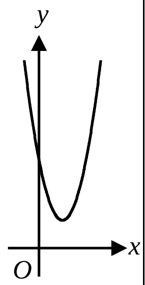
VII. The shape of second degree function $y = ax^2 + bx + c\ (a \neq 0)$ is a parabola, symmetric on both sides of the symmtry line $x = -\frac{b}{2a}$, with $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$ as its vertex. There are 3 situations for the case $a > 0$ (also 3 situations for the case $a < 0$):

(i) when $x < -\frac{b}{2a}$, y decreases (increases for $a < 0$) as the value of x increases;

(ii) when $x = -\frac{b}{2a}$, y has a minimum value (maximum value for $a < 0$) of $\frac{4ac - b^2}{4a}$;

(iii) when $x > -\frac{b}{2a}$, y increases (decreases for $a < 0$) as the value of x increases

VIII. The following table tabulates a comparison between a general second degree function, a quadratic equation in one variable and a quadratic inequality in one variable under different values of the discriminant:

Discriminant $\Delta = b^2 - 4ac$		$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
Graph of Second degree function $y = ax^2 + bx + c$ $(a > 0)$				
Roots of Quadratic Equation in one variable $ax^2 + bx + c = 0$ $(a \neq 0)$		Two unequal roots $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $(x_1 < x_2)$	Two equal roots $x_1 = x_2 = -\dfrac{b}{2a}$	No real root
Quadractic inequality	$ax^2 + bx + c > 0$ $(a > 0)$	$x < x_1$ or $x > x_2$	All real numbers not equal to $-\dfrac{b}{2a}$	All real numbers
	$ax^2 + bx + c < 0$ $(a > 0)$	$x_1 < x < x_2$	Empty set	Empty set

IX. The solution set of a group of simultaneous linear inequalities in one variable is the solution set common to all the solutions of the constituent inequalities.

X. When $a > 0$,

(i) the solution set of $|x| < a$ is $-a < x < a$;

(ii) the solution set of $|x| > a$ is $x > a$, or $x < -a$.

Revision Exercise 14

1. On a rectangular coordinate plane,
   (1) For any point in the first quadrant, what is the sign of its *x* coordinate and what is the sign of its *y* coordinate?
   (2) For any point in the fourth quadrant, what is the sign of its *x* coordinate and what is the sign of its *y* coordinate?
   (3) If the *y* coordinate of a point is negative and its *x* coordinate is positive, which quadrant does the point lie in?
   (4) If the *y* coordinate and the *x* coordinate of a point are both negative, which quadrant does the point lie in?
2. If $P_1(0,\ 2)$, $P_2(8, -4)$, $P_3(5, -8)$, $P_4(-3, -2)$ are the four vertices of a quadrilateral, (i) is it a parallelegram? (ii) is it a rectengle? (iii) draw this quadrilateral on the coordinate plane.
3. The Volume of a cylindrical cone is given by the formula $V = \dfrac{1}{3}\pi r^2 h$, where *V* is the volume, *r* is the radius of the circular base and *h* is its height.
   (1) when *r* is constant, what is the relationship between *V* and *h*?
   (2) when *h* is constant, what is the relationship between *V* and the area of the base *A* (where $A = \pi r^2$)?
   (3) when *V* is constant, what is the relationship between *A* and *h*?
   (4) when *V* is constant, is the relationship between *r* and *h* inversely proportional, and why?
   (5) when $r = 1$, draw the graph that show the change of value of *V* in relation to *h*.
   (6) when $V = 6$, draw the graph that shows the change of value of *A* in relation to *h*.

4. The volume of water Q(L) that flows out of a pipe in relation to time t (second) is given by the function $Q = kt$, where k is a constant.
   (1) what is the relationship between Q and t?
   (2) It is known that in 5 seconds of time, the volume of water that flows out is 120 L. Find the value of the scale factor k;
   (3) Using the result in (2), find the volume of water that flows out of the pipe in 8.5 seconds;
   (4) Also using the result in (2), find the time required for 320 L of water to flow out from the pipe.

5. (1) If x and y are directly proportional; y and z are directly proportional, then what is the relationship between x and z?
   (2) If x and y are inversely proportional; y and z are inversely proportional, then what is the relationship between x and z?
   (3) If x and y are directly proportional; y and z are inversely proportional, then what is the relationship between x and z?

6. Given two functions $y_1 = k_1x + b_1$, $y_2 = k_2x + b_2$.
   (1) If $k_1 = k_2$ and $b_1 \neq b_2$, what is the relationship between the graphs of these two functions?
   (2) If $k_1 \neq k_2$ and $b_1 = b_2$, what is the relationship between the graphs of these two functions?
   (3) If $k_1 = k_2$ and $b_1 = b_2$, what is the relationship between the graphs of these two functions?

7. If the coordinates of points (x_1, y_1), (x_2, y_2) satisfy the function $y = kx + b$, find the function.

8. Given a function $y = 3x - 15$.
   (1) Draw the graph of this function.
   (2) From the graph, find the value of x, when the function is
      (i) greater than zero;
      (ii) less than zero;
      (iii) equal to zero.
   (3) From the above results, is it possible to derive the relationship between the following:
      (i) a first degree function;
      (ii) a linear equation in one varable, and;
      (iii) a linear inequality in one varable?

9. From the graph of $y = 2x - 3$, find the following:
   (1) when $y = 2$, what is the value of x?
   (2) when $x < 0$, what is the range of values of y?
   (3) when $y > 3$, what is the domain of values of x?
   (4) when $y < 5$, what is the domain of values of x?

10. Find the parabola that passes through the three points $A(0, 1)$, $B(-1, 1)$ and $C(1, -1)$, with the symmetry line parallel to the y axis. Also find its vertex and symmetry line.

11. Given the symmetry line is parallel to the y *axis*, and that the parabola has vertex at (2, 3), and passes through the point (3, 1), it is required to find the parabola.

12. Find the maximum or minimum of the following function, and find the corresponding value of the independent variable.
   (1) $y = 2x^2 + 5$; (2) $y = (x-3)^2 - 2$;
   (3) $y = ax^2 - bx$ (discussion recommended);
   (4) $y = (a + x)(b - x)$.

13. Solve x from the inequality $ax + b > cx + d$ (discussion recommended).

14. Solve the following simultaneous inequalities:
   (1) $\begin{cases} 1 - \dfrac{x+1}{2} \leq 2 - \dfrac{x+2}{3} \\ x(x-1) \geq (x+3)(x-3) \end{cases}$ (2) $\begin{cases} 3 + x < 4 + 2x \\ 5x - 3 < 4x - 1 \\ 7 + 2x > 6 + 3x \end{cases}$

15. Solve x from the inequality (discussion recommended)
   (1) $|x - a| < b$; (2) $|x - a| > b$.

16. Solve the simultaneous inequalities

$$\begin{cases} x(x^2+1) \geq (x+1)(x^2-x+1) \\ \quad 1-2x > 3(x-9) \end{cases}$$

17. Solve the inequality $0 < x^2 - x - 2 < 4$.

This chapter is translated to English by courtesy of Mr. Hyman Lam and reviewed by courtesy of Mr SIN Wing Sang Edward.