International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2001.**

1. On the plane is a triangle with red vertices and a triangle with blue vertices. O is a point inside both triangles such that the distance from O to any red vertex is less than the distance from O to any blue vertex. Can the three red vertices and the three blue vertices all lie on the same circle?

2. Do there exist positive integers $a_1 < a_2 < \cdots < a_{100}$ such that for $2 \leq k \leq 100$, the least common multiple of a_{k-1} and a_k is greater than the least common multiple of a_k and a_{k+1}?

3. An 8×8 array consists of the numbers 1, 2, ..., 64. Consecutive numbers are adjacent along a row or a column. What is the minimum value of the sum of the numbers along a diagonal?

4. Let F_1 be an arbitrary convex quadrilateral. For $k \geq 2$, F_k is obtained by cutting F_{k-1} into two pieces along one of its diagonals, flipping one piece over and then glueing them back together along the same diagonal. What is the maximum number of non-congruent quadrilaterals in the sequence $\{F_k\}$?

5. Let a and d be positive integers. For any positive integer n, the number $a + nd$ contains a block of consecutive digits which constitute the number n. Prove that d is a power of 10.

6. In a row are 23 boxes such that for $1 \leq k \leq 23$, there is a box containing exactly k balls. In one move, we can double the number of balls in any box by taking balls from another box which has more. Is it always possible to end up with exactly k balls in the k-th box for $1 \leq k \leq 23$?

7. The vertices of a triangle have coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3). For any integers h and k, not both 0, the triangle whose vertices have coordinates (x_1+h, y_1+k), (x_2+h, y_1+k) and $(x_3 + h, y_3 + k)$ has no common interior points with the original triangle.

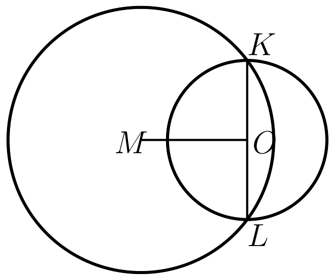
   (a) Is it possible for the area of this triangle to be greater than $\frac{1}{2}$?

   (b) What is the maximum area of this triangle?

Note: The problems are worth 4, 5, 6, 6, 7, 7 and 3+6 points respectively.

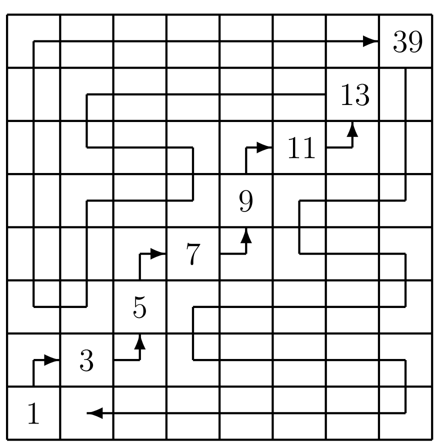
Solution to Senior A-Level Fall 2001

1. Suppose all six vertices lie on a circle with centre M. Let the line through O perpendicular to OM cut the circle at K and L. Since M is inside the triangle with red vertices, at least one red vertex lies on the minor arc KL and at least one red vertex lies on the major arc KL. The same is true of the blue vertices. However, every point on the minor arc KL is inside the circle with diameter KL, so that its distance from O is at most OK. On the other hand, every point on the major arc KL is outside the circle with diameter KL, so that its distance from O is at least OK. This is a contradiction.



2. We use a_n to denote the n-th term, even though its value may be modified along the way. In step 1, we set $a_{99} = 9$ and $a_{100} = 10$, with $\text{lcm}(a_{99}, a_{100}) = 90$. In step $k > 1$, define $a_{100-k} = 10a_{101-k} - 1$ and then redefine a_n for each $n > 100 - k$ to be 10 times its former value. Hence in step 2, we define $a_{98} = 10a_{99} - 1 = 89$. We also redefine $a_{99} = 90$ and $a_{100} = 100$. We have $\text{lcm}(a_{98}, a_{99}) = 8010 > 900 = \text{lcm}(a_{99}, a_{100})$. We continue until step 99 has been completed. Note that once we have $\text{lcm}(a_{n-1}, a_n) > \text{lcm}(a_n, a_{n+1})$, this remains true thereafter since in all subsequent modifications, each of a_{n-1}, a_n and a_{n+1} is multiplied by the same number. We only have to check this inequality when a_{n-1} is first introduced. At this point, $a_{n-1} = a_n - 1 = a_{n+1} - 11$. Now $10a_{n-1} > a_{n-1} + 11 = a_{n+1}$ since $a_{n-1} > 1$. Hence $\text{lcm}(a_{n-1}, a_n) = a_{n-1}a_n > \frac{1}{10}a_n a_{n+1} = \text{lcm}(a_n, a_{n+1})$.

3. Since consecutive numbers occupy sqaures of opposite colours, we may assume that all numbers on black squares are odd and all numbers on white squares are even. The diagram below shows that the sum may be as small as 1+3+5+7+9+11+13+39=88.



Suppose it is possible to improve on this. Clearly, the diagonal in question should contain odd numbers, and the largest would have to be at most 37. Once this number is put down, we must remain on the same side of this diagonal. There are exactly 16 black squares and 12 white squares on each side. Hence that largest number is 37 and only one square on the largely empty side of the diagonal has been filled. However, there are 13 odd numbers from 38 to 64 but we have at most 12 white squares to accommodate them. Hence improvement over 88 is impossible.

4. Let $F_1 = ABC_1D_1$ and let $F_2 = ABC_1D_2$ be obtained from F_1 by reflecting D_1 to D_2 across the perpendicular bisector of AC_1. Reflecting alternately across the two diagonals, we obtain $F_3 = ABC_2D_2$, $F_4 = ABC_2D_3$, $F_5 = ABC_3D_3$, $F_6 = ABC_3D_4$ and $F_7 = ABC_4D_4$. This sequence of transformations permutes the sides while preserving the sum of the opposite angles. We have $\angle ABC_1 + \angle C_1D_1A = \angle ABC_4 + \angle C_4D_4A$, $BC_1 = BC_4$, $C_1D_1 = C_4D_4$ and $D_1A = D_4A$. If $AC_1 > AC_4$, then $\angle ABC_1 > \angle ABC_4$ and $\angle C_1D_1A > \angle C_4D_4A$. We also have a contradiction if $AC_1 < AC_4$. Hence $AC_1 = AC_4$ and it follows that F_1 and F_7 are congruent. Thus the sequence $\{F_k\}$ consists of at most six non-congruent quadrilaterals. If F_1 has sides of distinct lengths and the sum of neither pair of opposite angles is $180°$, we indeed have six non-congruent quadrilaterals.

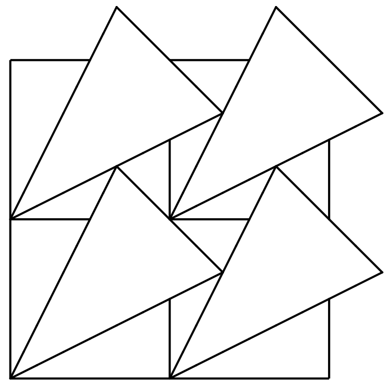
5. Let the number of digits of d be k, and that of a be m. Consider the term $a + 10^t d$ where t is an integer such that $t > \max\{k, m\}$. It must contain a 1 followed by at least m zeros, so that $k > m$. The next term $a + (10^t + 1)d$ must contain two 1's separated by exactly $t - 1$ zeros. Since $t > k$, this can only happen if the first digit of d is 1 and the remaining digits are 0's, which means that d is a power of 10.

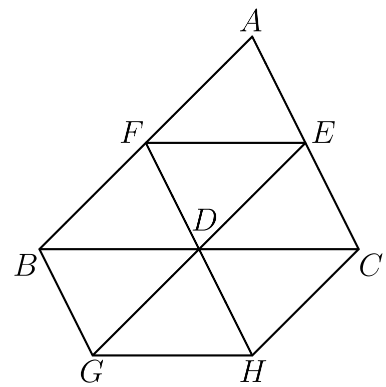
6. We shall prove by induction on the number n of boxes that the task is always possible. This is clearly true for $n = 1$. Suppose it is true for some $n \geq 1$. Consider the next case where we have $n + 1$ boxes. Line up the boxes from left to right in increasing order of the number of balls in them, without regard to the box numbers. Transfer balls from each box to the next one to its left, starting with the rightmost one which contains $n + 1$ balls.

1	2	3	$\cdots$	$n-1$	n	$n+1$
					$2n$	1
				$2n-2$	$n+1$	
			$\cdots$	n		
		6	$\cdots$			
	4	4				
2	3					
2	3	4	$\cdots$	n	$n+1$	1

This sequence of moves results in a cyclic permutation of the numbers of the balls. We perform this a number of times until the $(n+1)$-st box contains $n + 1$ balls. The rest of the boxes can be sorted out by the induction hypothesis.

7. (a) The tiling on the left of the figure below shows that the area of the triangle may be $\frac{2}{3}$. The coordinates of the vertices of a copy of the triangle are (0,0), $(\frac{4}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{4}{3})$.





(b) Let ABC be any triangle with the desired properties. Let D, E and F be the midpoints of BC, CA and AB respectively. Extend ED to G and FD to H so that $ED = DG$ and $FD = DH$, as shown on the right of the figure above. We claim that integral translates of the hexagon $BGHCEF$ do not have common interior points. It will then follow that its area is at most 1, and that the area of ABC is at most $\frac{2}{3}$. This maximum is attained by the example in (a). Suppose to the contrary that $BGHCEF$ has a common point with an integral translate $B'G'H'C'E'F'$. We may assume that either E' or F' is inside the quadrilateral $BGHC$. There are three cases. If E' is inside triangle DBG, then B will be inside the integral translate $A'B'C'$ of ABC. Similarly, if F' is inside triangle DCH, then C will be inside $A'B'C'$. Finally, if either E' or F' is inside triangle DGH, then A' will be inside ABC. Since we have a contradiction in each of the three cases, the claim is justified.