International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2002.**

1. Several diagonals of a convex polygon of 2002 sides are drawn with no two intersecting inside the polygon, partitioning it into 2000 triangles. Is it possible for exactly half of these triangles to have diagonals for all three sides?

2. John and Mary each chose a positive integer and told only Bill. Bill told them that either the sum or the product of their numbers was 2002. John said that he could not determine what number Mary had chosen. Having heard that, Mary said she still could not determine what number John had chosen. What was the number chosen by Mary?

3. In a contest, at least two-thirds of the problems were such that each was not solved by at least two-thirds of the students. On the other hand, at least two-thirds of the students were such that each solved at least two-thirds of the problems.

   (a) Is this possible?

   (b) Will the answer be the same if two-thirds is replaced by three-fourths everywhere?

   (c) Will the answer be the same if two-thirds is replaced by seven-tenths everywhere?

4. A game is played with 2002 cards, with the numbers $1, 2, 3, \ldots, 2002$ written on them. Two players alternately select one card at a time until all have been taken. The winner is the player for whom the last digit of the sum of the numbers on the cards chosen is the largest. Which of the two players can win regardless of how the other plays, and what is the winning strategy?

5. Three straight lines are drawn through a given point inside an angle, cutting each arm of the angle at three points. Is it possible that on each arm, the point in the middle is exactly halfway between the other two?

Note: The problems are worth 4, 5, 1+2+2, 5 and 5 points respectively.

Solution to Junior O-Level Fall 2002

1. Each triangle can have at most two of the 2002 sides of the polygon as its sides. Hence at least 1001 of the 2000 triangles have at least one side of the polygon as a side, leaving at most 999 triangles which may have diagonals for all three sides.

2. John's number must be a divisor of 2002 as otherwise he would know that 2002 is the sum, and can easily determine Mary's number. However, it cannot be 2002 as otherwise he would know that 2002 is the product and hence John's number would be 1. The same can be said about Mary's number. Moreover, if it is less than 1001, then she would know that 2002 cannot be the sum since John's number is at most 1001. Hence Mary's number must be 1001.

3. (a) This is possible. We may have 3 students and 3 problems. One student solved problems 1 and 3 but not 2, while another solved problems 1 and 2 but not 3, while the last student did not solve problem 2 or 3.

   (b) A student-problem combination is said to be positive if the student solved the problem, and negative otherwise. If at least three-quarters of the students solved at least three-quarters of the problems, then at least nine-sixteenth of the combinations were positive. On the other hand, if at least three-quarters of the problems were not solved by at least three-quarters of the studetns, then at least nine-sixteenth of the combinations were negative. This is impossible since $\frac{9}{16}+\frac{9}{16}>1$.

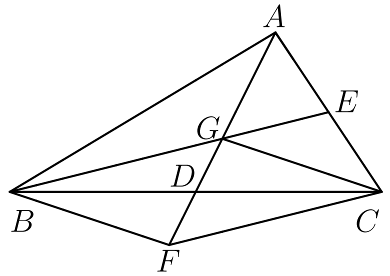
   (c) Let the fraction of students who solved at least $\frac{7}{10}$ of the problems be g and the fractional of the problems not solved by at least $\frac{7}{10}$ of the students be h. A combination of such a student and such a problem is said to be special. At least $g(\frac{7}{10}-(1-h))$ of the combinations are positive special combinations, and at least $h(\frac{7}{10}-(1-g))$ are negative special combinations. Hence $gh \geq g(h-\frac{3}{10})+h(g-\frac{3}{10})$, which is equivalent to

   $$9 \geq (10g-3)(10h-3).$$

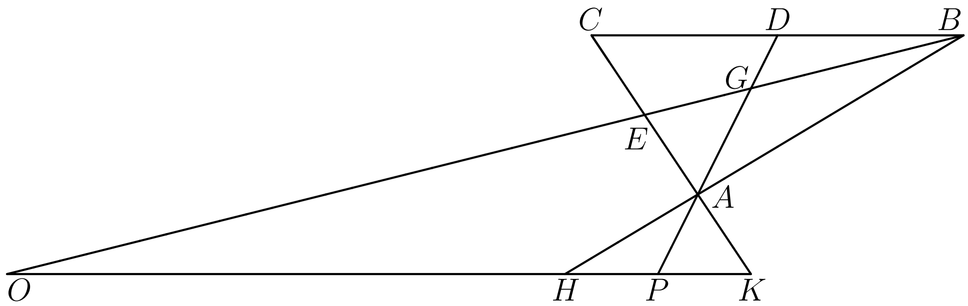
   If $g \geq \frac{7}{10}$ and $h \geq \frac{7}{10}$, then $9 \geq 16$, and we have a contradiction.

4. We can replace each card by one labelled with only the last digit of the original number. A set is defined as 10 cards with distinct labels. There are 200 sets plus two extra cards labelled with 1 and 2 respectively. The first player wins by taking a card labelled with 2, and thereafter match what the second player takes. The game ends when the second player takes the last card, which is labelled with 1. The first player has taken 100 sets plus a card labelled with 2. Hence the last digit of the sum of the numbers tajen is 2. The second player has taken 100 sets plus a card labelled with 1. Hence the last digit of the sum of the numbers taken is 1. This is a win for the first player.

5. Let G be any point on the median AD of triangle ABC. Let the extension of BG intersect CA at E. We claim that $BG > GE$. Extend AD to F so that $GD = DF$. Since the diagonals of $AFCG$ bisect each other, it is a parallelogram. Hence GE is parallel to FC. so that triangles AGE and AFC are similar. Since $AG < AF$, we have $GE < FC = BG$ as claimed.



Now let A be a point inside an angle with vertex O. Let the three lines through A cut one arm of the angle at H, P and K, and the other arm at B, G and E, as shown in the diagram below. Suppose P is the midpoint of HK. Through B, draw a line parallel to OK, cutting the extension of PG at D and the extension of KE at C. Then triangles BAD and HAP are similar, as are triangles CAD and KAP. Hence $\frac{BD}{HP} = \frac{AD}{AP} = \frac{CD}{KP}$. Since $HP = KP$, we have $BD = CD$. It follows from the claim about that $BG > GE$, so that G cannot be the midpoint of BE.



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   (a) Is this possible?

   (b) Will the answer be the same if two-thirds is replaced by three-fourths everywhere?

   (c) Will the answer be the same if two-thirds is replaced by seven-tenths everywhere?

3. The plane is partitioned into regions by several straight line, no two of which are parallel. Prove that a point lies in an umbounded region if and only if there exists another point such that these two points are on opposite sides of each of the lines.

4. Let x, y and z be any positive real numbers less than $\frac{\pi}{2}$. Prove that $\frac{(x\cos x+y\cos y+z\cos z}{x+y+z} \leq \frac{\cos x+\cos y+\cos z}{3}$.

5. In an infinite sequence $\{x_n\}$ of positive integers, x_{n+1} is the sum of x_n and a non-zero digit of x_n for $n \geq 1$. Prove that x_n is even for some $n \geq 1$.

Note: The problems are worth 4, 1+2+2, 5, 5 and 5 points respectively.

Solution to Senior O-Level Fall 2002

1. John's number must be a divisor of 2002 as otherwise he would know that 2002 is the sum, and can easily determine Mary's number. However, it cannot be 2002 as otherwise he would know that 2002 is the product and hence John's number would be 1. The same can be said about Mary's number. Moreover, if it is less than 1001, then she would know that 2002 cannot be the sum since John's number is at most 1001. Hence Mary's number must be 1001.

2. (a) This is possible. We may have 3 students and 3 problems. One student solved problems 1 and 3 but not 2, while another solved problems 1 and 2 but not 3, while the last student did not solve problem 2 or 3.

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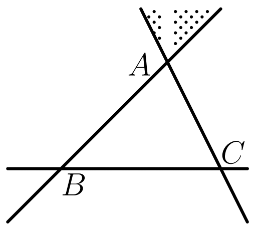
   (c) Let the fraction of students who solved at least $\frac{7}{10}$ of the problems be g and the fractional of the problems not solved by at least $\frac{7}{10}$ of the students be h. A combination of such a student and such a problem is said to be special. At least $g(\frac{7}{10}-(1-h))$ of the combinations are positive special combinations, and at least $h(\frac{7}{10}-(1-g))$ are negative special combinations. Hence $gh \geq g(h-\frac{3}{10})+h(g-\frac{3}{10})$, which is equivalent to

$$9 \geq (10g-3)(10h-3).$$

   If $g \geq \frac{7}{10}$ and $h \geq \frac{7}{10}$, then $9 \geq 16$, and we have a contradiction.

3. Draw a circle large enough to contain all points of intersection of the lines. Then an unbounded region becomes one with a curvilinear side. Let the arc AB be a side of such a region. Then no other line terminates between A and B. We claim that the arc $A'B'$ is also a side of such a region, where AA' and BB' are two of the lines. Suppose a third line CC' terminates at C' between A' and B'. We know that C is not between A and B. If it is between A and B', then CC' will not intersect AA' inside the circle. If it is between B and A', then CC' will not intersect BB' inside the circle. Finally, if it is between A' and B', then CC' will not intersect either AA' or BB' inside the circle. This justifies the claim. For any point inside the region with the arc AB as a side, choose any point inside the region with the arc $A'B'$ as a side. For any other line CC', if both C and C' are between A and B', or both between B and A', then CC' will not intersect either AA' or BB' inside the circle. Hence one of them is between A and B' while the other is between B and A'. Then the two chosen points are on opposite sides of CC'.

For any finite region, we can choose three sides so that the region is contained in the triangle ABC they form. Choose any point in the original region. For any point to be on the opposite side to it of AB and AC, it must lie inside the shaded region as shown. However, the two points will be on the same side of BC.



4. We may assume that $x \leq y \leq z$. Since cosine is a decreawing function on the open interval $(0, \frac{\pi}{2})$, $\cos x \geq \cos y \geq \cos z$. By the Rearrangement Inequality,

$$z \cos x + x \cos y + y \cos z \geq x \cos x + y \cos y + z \cos z,$$

$$y \cos x + z \cos y + x \cos z \geq x \cos x + y \cos y + z \cos z$$

and

$$x \cos x + y \cos y + z \cos z \geq x \cos x + y \cos y + z \cos z.$$

Adding these inequalities yields the desired result.

5. Suppose x_n is odd for all $n \geq 1$. Then x_1 must have an even digit which is not in the last place. Let k be its first even digit. Since the sequence increases without bounds, this digit k becomes $k+1$ in the term x_m for some $m > 1$. All digits before this $k+1$ remain unchanged from x_1 and therefore odd. All digits after it are zero's except the last digit whcih is odd. Hence x_{m+1} will be even, which is a contradiction.