International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2002.**

1. Arrange around a circle are 2002 distinct numbers such that the difference between any two adjacent numbers is 2 or 3. What can the maximal difference between any two of these numbers?

2. All 19999 species of plants existing in Russia are numbered consecutively by the integers from 2 to 20000. For any pair of species, the greatest common divisor of their numbers was recorded, but the numbers themselves were lost as the result of a computer error. Is it possible to recover the number of each of the species from the recorded data?

3. The vertices of a polygon of 50 sides divide a circle into 50 arcs, whose lengths are 1, 2, 3,. . . ., 50, in some order. The difference in length of the arcs corresponding to opposite sides of the polygon is 25. Prove that the polygom has two parallel sides.

4. P is a point inside triangle ABC such that $\angle ABP = \angle ACP$ and $\angle CBP = \angle CAP$. Prove that P is the orthocentre of the triangle.

5. A convex polygon of N sides is divided into triangles by diagonals whichi do not intersect inside the polygon. The triangles are painted black and white so that any two triangles with a common side are painted in different colours. For each N, determine the maximal difference between the number of black and the number of white triangles.

6. Let n and k be positive integers. On each card in a deck is written one of the numbers 1, 2, . . . , n. The sum of the numbers on all the cards is $k(n!)$. Prove that the deck can be divided into k stacks such that the sum of the numbers on the cards of each stack is $n!$.

7. An electrical network consists of n^2 nodes in an $n \times n$ array. Each node is connected by wire to all adjacent nodes in the same row or the same column. Some of the wires may be burned out. In one testing, one may choose any two nodes and check if there is a chain of intact wires joining the chosen nodes. Actually this is the case for any two nodes, but this requires verification. What is the minimum number of testing needed if

   (a) $n = 4$;

   (b) $n = 6$?

Note: The problems are worth 4, 5, 6, 6, 7, 9 and 5+5 points respectively.

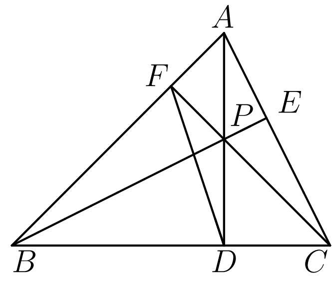
Solution to Junior A-Level Fall 2002

1. We may assume that the smallest of the 2002 numbers is 0. Then the largest is at most $1001 \times 3 = 3003$ if it is diametrically opposite to 0, and all differences between adjacent pairs are 3. However, this means that the 2002 numbers will not be distinct. Hence the largest is at most 3002, and this can be attained with 0, 3, 6, ..., 3000, 3002, 2999, 2996, ..., 2.

2. Consider species number $8192 = 2^{13}$ and species number $16384 = 2^{14}$. For any other species number n, its greatest common divisor with 8192 or 16384 is a power of 2. If the two values are different, it can be happen if $\gcd(n, 8192) = 2^{13}$ while $\gcd(n, 16384) = 2^{14}$. However, the smallest value of $n > 16384$ for which $\gcd(n, 16384) = 2^{14}$ is $2^{15} = 32768$, which exceeds 20000. Hence species number 8192 and species number 16384 are now indistinguishable.

3. For each arc A, denote the opposite arc by A'. The left sum of A is the total length of the arcs between A and A' in clockwise order, and the right sum of A is the total length of the arcs between A' and A in clockwise order. If the left sum of A is equal to the right sum of A, then the sides corresponding to A and A' are parallel. Suppose the two sums are not equal. We may assume that the left sum of A is greater than the right sum of A, so tht the difference between the left sum and the right sum is positive. Moreover, it is always an even multiple of 25 because it is the sum of 24 copies of ± 25. We now consider this difference for each arc between A and A' in clockwise order. Since the left sum of A' is the right sum of A, it is less than the left sum of A which is the right sum of A'. Since the difference has changed from positive to negative, it must have hit 0 somewhere along the way. Thus we always have a pair of opposite sides which are parallel.

4. Let AP, BP and CP cut the opposite sides at D, E and F respectively. Note that we have $\angle AEP = 180° - \angle CAP - \angle APE = 180° - \angle CBP - \angle BPD = \angle BDP$. Similarly, $\angle BFP = \angle CEP$. Since $\angle BFP + \angle BDP = \angle AEP + \angle CEP = 180°$, $BDPF$ is cyclic. Hence $\angle FDP = \angle ABP = \angle ACP$, so that $ACDF$ is also cyclic. It follows that $\angle CDP = \angle AFP$. Now $360° = \angle CDP + \angle AFP + \angle BDP + \angle BFP = 2\angle CDP + 180°$. Hence $\angle CDP = \angle AFP = 90°$. It follows that AD and CF are altitudes, and P is indeed the orthocentre of triangle ABC.



5. Construct a graph as follows. Represent each triangle by a node, and join two nodes by an edge if the triangles they represent share a common side. Each node is of degree at most 3, since each triangle has at most 3 neighbours. Since all vertices of the triangles are on the perimeter of the polygon, our graph is a tree. We paint the nodes black and white so that adjacent nodes have different colours. We may assume that there are at least as many black nodes as white nodes. Let $f(N)$ denote the maximum difference between their numbers for a polygon with N sides. We have $f(3) = 1$, $f(4) = 0$ and $f(5) = 1$. In each case, the optimal

graph has a black node of degree 1. To any optimal tree, add 2 black nodes and 1 white node, with the black nodes adjacent only to that white node and the white node adjacent to an existing black node of degree 1. Then $f(N+3) \geq f(N)+1$ for $N \geq 3$. It follows that $f(3k) \geq k$, $f(3k+1) \geq k-1$ and $f(3k+2) \geq k$ for $k \geq 1$. We now prove that these bounds are in fact sharp. Let $N = 3k$. Then there are $3k-2$ nodes. Suppose the optimal graph has w white nodes, so that their total degree is at most $3w$. Then there are $3k-2-w$ black nodes. Now at least $w-1$ of them must have degree at least 2. Hence their total degree is at least $3k-2-w+(w-1) = 3k-3$. Since the total degree of the white nodes must be equal to the total degree of the black nodes, we have $3w \geq 3k-3$ or $w \geq k-1$. Hence the difference is $3k-2-w-w \leq 3k-2-2(k-1) = k$. For $N = 3k+1$, the inequality is $3w \geq 3k-2$ or $w \geq k$. Hence the difference is $3k-1-w-w \leq 3k-1-2k = k-1$. For $N = 3k+2$, the inequality is $3w \geq 3k-1$ or $w \geq k$. Hence the difference is $3k-w-w \leq 3k-2k = k$.

6. We claim that any set of n integers contains a subset whose sum is a multiple of n. Let the set be $\{a_1, a_2, \ldots, a_n\}$. For $1 \leq m \leq n$, define $b_m = a_1 + a_2 + \cdots + a_m$. If $b_m \equiv 0 \pmod{n}$ for any m, we have nothing further to prove. Otherwise, by the Pigteonhole Principle, we must have $b_i \equiv b_j \pmod{n}$ for some i and j where $1 \leq i < j \leq n$. Then $a_{i+1} + a_{i+2} + \cdots + a_j \equiv 0 \pmod{n}$. This justifies the claim. We now solve the original problem by induction on n. The basis is trivial. Suppose the result is true for $n-1$. Consider any collection of 1, 2, …, n with sum $k(n!)$. First put each n in an envelope and label the envelope 1. Now use the claim to pick up to n cards whose sum is tn for some integer t, and put them inside an envelope. Since each card is at most $n-1$, we have $t \leq n-1$. Label the envelope t. Eventually, we are left with $r < n$ cards. If $r = 0$, nothing else is done. If $r > 0$, then the sum of these r cards is also a multiple of n since so is the sum of all the cards. Label the emvelope t if this sum is tn. Now we have a collection of envelopes each labelled 1, 2, …, $n-1$, and the sum of all the labels is $k((n-1)!)$. By the induction hypothesis, the envelopes can be divided into k stacks such that the sum of the labels is $(n-1)!$. For each stack, open all the envelopes and put all the cards inside into a stack. The sum of the numbers on the cards in each stack is $n!$ as desired.

7. Let $n = 2k$. We must perform at least $2k^2$ tests since each node must be involved in at least one test. We claim that this lower bound can be attained. Labels the nodes (i,j) where $1 \leq i \leq 2k$ and $1 \leq j \leq 2k$. On the main diagonal, we perform the tests between (i,i) and $(k+i, k+i)$ for $1 \leq i \leq k$. Off the main diagonal, we perform the tests between (i,j) and (j,i) for all $i \neq j$. The test results indicate that (1,1) is connected to $(k+1, k+1)$. Moreover, they are connected by at least two paths which are symmetrical with respect to the main diagonal. Thus these two paths form a cycle which encloses the nodes (i,i) but leaves on the outside the nodes $(k+i, k+i)$, where $2 \leq i \leq k$. Since (i,i) and $(k+i, k+i)$ are connected, they must be connected to the cycle above. It follows that all nodes on the main diagonal are connected to one another. Since each node off the main diagonal is connected to a node on the other side, all nodes are indeed connected.

International Mathematics
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Senior A-Level Paper **Fall 2002.**

1. All 19999 species of plants existing in Russia are numbered consecutively by the integers from 2 to 20000. For any pair of species, the greatest common divisor of their numbers was recorded, but the numbers themselves were lost as the result of a computer error. Is it possible to recover the number of each of the species from the recorded data?

2. A planar cross-section of a cube is a pentagon. Prove that the length of at least one side of the pentagon differs from 1 meter by at least 20 centimeters.

3. A convex polygon of N sides is divided into triangles by diagonals whichi do not intersect inside the polygon. The triangles are painted black and white so that any two triangles with a common side are painted in different colours. For each N, determine the maximal difference between the number of black and the number of white triangles.

4. Let n and k be positive integers. On each card in a deck is written one of the numbers 1, 2, $\dots$, n. The sum of the numbers on all the cards is $k(n!)$. Prove that the deck can be divided into k stacks such that the sum of the numbers on the cards of each stack is $n!$.

5. Two circles intersect at points A and B. Through the point B, a straight line is drawn, intersecting the first again at K and the second circle again at M. A line parallel to AM is tangent to the first circle at Q. The line AQ intersects the second circle again at R.

   (a) Prove that the tangent to the second circle at R is parallel to AK.

   (b) Prove that these two tangents are concurrent with KM.

6. The first two terms of a sequence are 1 and 2 respectively. Each subsequent term is the smallest positive integer which has not yet occurred in the sequence and is not relatively prime to the preceding term of the sequence. Prove that all positive integers appear in this sequence.

7. An electrical network consists of n^2 nodes in an $n \times n$ array. Each node is connected by wire to all adjacent nodes in the same row or the same column. Some of the wires may be burned out. In one testing, one may choose any two nodes and check if there is a chain of intact wires joining the chosen nodes. Actually this is the case for any two nodes, but this requires verification. What is the minimum number of testing needed if

   (a) $n = 4$;

   (b) $n = 8$?

Note: The problems are worth 4, 6, 6, 8, 4+4, 8 and 5+5 points respectively.

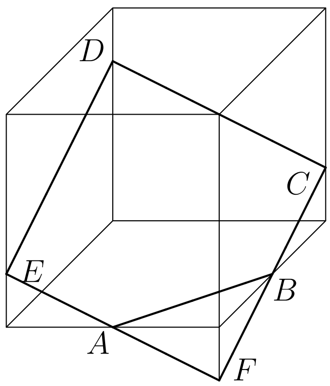
Solution to Senior A-Level Fall 2002

1. Consider species number $8192 = 2^{13}$ and species number $16384 = 2^{14}$. For any other species number n, its greatest common divisor with 8192 or 16384 is a power of 2. If the two values are different, it can be happen if $\gcd(n, 8192) = 2^{13}$ while $\gcd(n, 16384) = 2^{14}$. However, the smallest value of $n > 16384$ for which $\gcd(n, 16384) = 2^{14}$ is $2^{15} = 32768$, which exceeds 20000. Hence species number 8192 and species number 16384 are now indistinguishable.

2. We may assume that the planar section misses the top of the cube. Then it intersects the four vertical edges at C, D, E and F as shown in the diagram below, where C, D and E are on the actual edges while F is below the bottom of the cube. Now CD and EF lie on two opposite faces of the cube. Hence they do not intersect. However, since they are also coplanar, they must be parallel to each other. Similarly, so are CF and DE. Let EF and CF intersect the bottom of the cube at A and B respectively, so that $ABCDE$ is the pentagonal cross-section. Then

$$\begin{aligned} 2\max\{CD, DE\} &\geq CD + DE \\ &= CB + BF + FA + AE \\ &> CB + BA + AE \\ &\geq 3\min\{CB, BA, AE\}. \end{aligned}$$

It follows that the ratio between the lengths of the longest and shortest sides of the pentagon is greater than $\frac{3}{2}$. However, if all five sides have lengths between 80 and 120 centimetres, the above ratio is at most $\frac{3}{2}$, which is a contradiction.

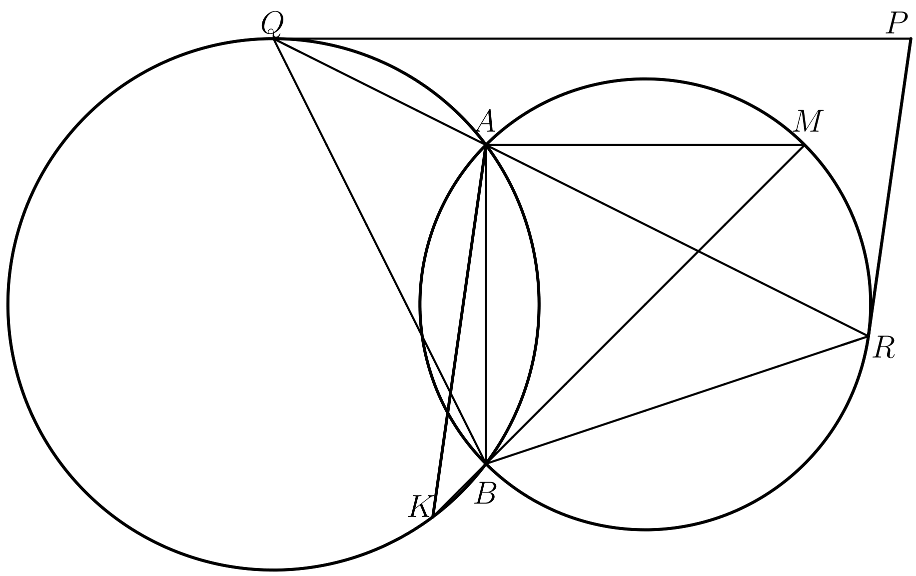


3. Construct a graph as follows. Represent each triangle by a node, and join two nodes by an edge if the triangles they represent share a common side. Each node is of degree at most 3, since each triangle has at most 3 neighbours. Since all vertices of the triangles are on the perimeter of the polygon, our graph is a tree. We paint the nodes black and white so that adjacent nodes have different colours. We may assume that there are at least as many black nodes as white nodes. Let $f(N)$ denote the maximum difference between their numbers for a polygon with N sides. We have $f(3) = 1$, $f(4) = 0$ and $f(5) = 1$. In each case, the optimal graph has a black node of degree 1. To any optimal tree, add 2 black nodes and 1 white node, with the black nodes adjacent only to that white node and the white node adjacent to

an existing black node of degree 1. Then $f(N+3) \geq f(N)+1$ for $N \geq 3$. It follows that $f(3k) \geq k$, $f(3k+1) \geq k-1$ and $f(3k+2) \geq k$ for $k \geq 1$. We now prove that these bounds are in fact sharp. Let $N = 3k$. Then there are $3k-2$ nodes. Suppose the optimal graph has w white nodes, so that their total degree is at most $3w$. Then there are $3k-2-w$ black nodes. Now at least $w-1$ of them must have degree at least 2. Hence their total degree is at least $3k-2-w+(w-1) = 3k-3$. Since the total degree of the white nodes must be equal to the total degree of the black nodes, we have $3w \geq 3k-3$ or $w \geq k-1$. Hence the difference is $3k-2-w-w \leq 3k-2-2(k-1) = k$. For $N = 3k+1$, the inequality is $3w \geq 3k-2$ or $w \geq k$. Hence the difference is $3k-1-w-w \leq 3k-1-2k = k-1$. For $N = 3k+2$, the inequality is $3w \geq 3k-1$ or $w \geq k$. Hence the difference is $3k-w-w \leq 3k-2k = k$.

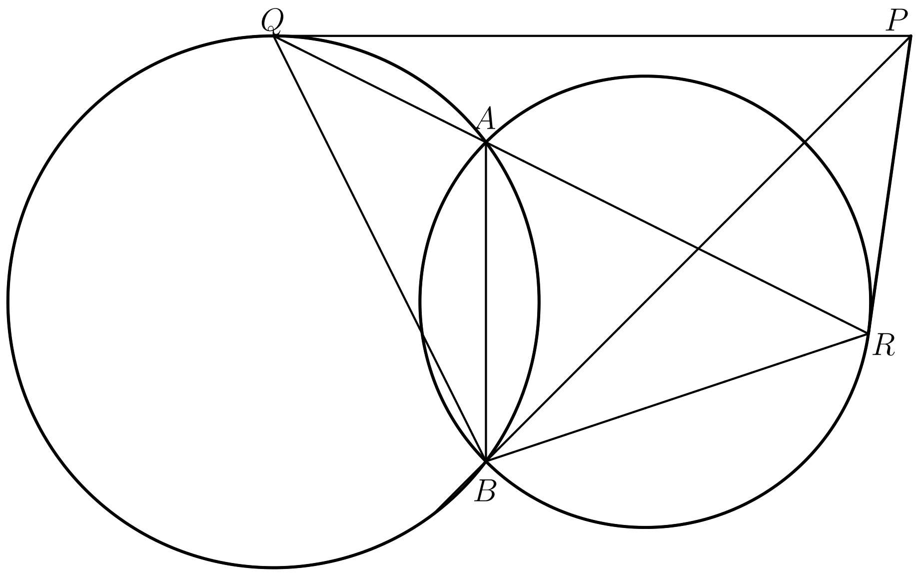
4. We claim that any set of n integers contains a subset whose sum is a multiple of n. Let the set be $\{a_1, a_2, \ldots, a_n\}$. For $1 \leq m \leq n$, define $b_m = a_1 + a_2 + \cdots + a_m$. If $b_m \equiv 0 \pmod{n}$ for any m, we have nothing further to prove. Otherwise, by the Pigteonhole Principle, we must have $b_i \equiv b_j \pmod{n}$ for some i and j where $1 \leq i < j \leq n$. Then $a_{i+1} + a_{i+2} + \cdots + a_j \equiv 0 \pmod{n}$. This justifies the claim. We now solve the original problem by induction on n. The basis is trivial. Suppose the result is true for $n-1$. Consider any collection of 1, 2, $\ldots$, n with sum $k(n!)$. First put each n in an envelope and label the envelope 1. Now use the claim to pick up to n cards whose sum is tn for some integer t, and put them inside an envelope. Since each card is at most $n-1$, we have $t \leq n-1$. Label the envelope t. Eventually, we are left with $r < n$ cards. If $r = 0$, nothing else is done. If $r > 0$, then the sum of these r cards is also a multiple of n since so is the sum of all the cards. Label the emvelope t if this sum is tn. Now we have a collection of envelopes each labelled 1, 2, $\ldots$, $n-1$, and the sum of all the labels is $k((n-1)!)$. By the induction hypothesis, the envelopes can be divided into k stacks such that the sum of the labels is $(n-1)!$. For each stack, open all the envelopes and put all the cards inside into a stack. The sum of the numbers on the cards in each stack is $n!$ as desired.

5. Denote the point of intersection of the two tangents by P.



(a) By Thales' Theorem, $\angle RBM = \angle RAM$. Since AM and QP are parallel, we have $\angle RAM = \angle RQP$. Since QP is tangent to the first circle, $\angle RQP = \angle QBA$. Similarly, $\angle ARP = \angle ABR = \angle ABM + \angle RAM = \angle ABM + \angle RAM = \angle QBM$. By Thales' Theorem, $\angle QAK = \angle QBK$. Hence $\angle QBM = 180° - \angle QBK = 180° - \angle QAK = \angle KAR$. From $\angle ARP = \angle KAR$, we conclude that AK and PR are parallel.

(b) We have $\angle QPR + \angle QBR = \angle QBA + \angle RBA = \angle QPR + \angle AQP + \angle ARP = 180°$. Hence $BQPR$ is cyclic so that $\angle PBQ = \angle PRQ = \angle MBQ$ from (a). Hence P lies on MB.



6. We first prove two lemmas, and then use them to establish the main result.
Lemma 1. At least one prime number divides infintely many terms of the sequence.
Proof:
Suppose this is false. Then there is a largest even tern N in the sequence. Consider all prime numbers less than N and all of their multiples. This is a finite set. Hence there is a term M of the sequence which exceeds all of them. Let p be the smallest prime divisor of M. Then $p > N$. The term after M must be $2p$. This contradicts the maximality assumption on N.
Lemma 2. Every prime number divides infintely many terms of the sequence.
Proof:
By Lemma 1, there exists a prime number p which divides infintely many terms of the sequence. Let q be any other prime number. For each positive integer n, consider the point at the sequence beyond which all terms are greater than pq^n. Some such term L must be divisible by p. Since L and pq^n are not relatively prime, and yet the term after L is greater than pq^n, this means that pq^n must have already appeared in the sequence. It follows that q also divides infintely many terms of the sequence.
Theorem. Every positive integer appears in the sequence.
Proof:
Let K be any integer greater than 1 and let p be one of its divisors. Consider the point at the sequence beyond which all terms are greater than K. Some such term H must be divisible by p. Since H and K are not relatively prime, and yet the term after H is greater than K, this means that K must have already appeared in the sequence.

7. Let $n = 2k$. We must perform at least $2k^2$ tests since each node must be involved in at least one test. We claim that this lower bound can be attained. Labels the nodes (i, j) where $1 \le i \le 2k$ and $1 \le j \le 2k$. On the main diagonal, we perform the tests between (i, i) and $(k + i, k + i)$ for $1 \le i \le k$. Off the main diagonal, we perform the tests between (i, j) and (j, i) for all $i \ne j$. The test results indicate that (1,1) is connected to $(k+1, k+1)$. Moreover, they are connected by at least two paths which are symmetrical with respect to the main diagonal. Thus these two paths form a cycle which encloses the nodes (i, i) but leaves on the outside the nodes $(k + i, k + i)$, where $2 \le i \le k$. Since (i, i) and $(k + i, k + i)$ are connected, they must be connected to the cycle above. It follows that all nodes on the main diagonal are connected to one another. Since each node off the main diagonal is connected to a node on the other side, all nodes are indeed connected.